申报	系列：教师系列 教学科研并重型
	专业：数学
	职称：教授

业绩成果材料

(申报人的业绩成果材料包括论文、科研项目、获奖以及其他成果等)

单　位（二级单位）：　数学与信息学院

姓　名　　李海绸

材料核对人：　　单位盖章：

核对时间：

华南农业大学制

目　录

一、教学研究业绩

1. 教学研究项目：关于省级教改项目“人工智能在数学教学辅助中的应用与探索—以数学专业核心课程《实变函数与泛函分析》为例”的立项通知及有关佐证材料.....3

2. 教学研究项目：关于校级教改项目“人工智能在数学教学辅助中的应用与探索—以数学专业核心课程《实变函数与泛函分析》为例”的立项通知及有关佐证材料.....8

二、科研项目

1. 主持：关于国家自然科学数学天元基金“2024 复分析及其应用专题讲习班”项目的立项通知及佐证材料.....10

2. 主持：关于国家自然科学数学天元基金“函数空间在 BMO－Teichmüller 理论上的应用”项目的立项通知及佐证材料13

3. 主持：关于广东省自然科学基金面上项目“拟凸域上的解析函数空间及其算子理论研究”的合同及佐证材料.....16

4. 主持：关于广东省科技厅科技特派员“智慧农业科技特派团”项目的立项通知及佐证材料.....18

5. 参与：关于国家自然科学基金面上项目“解析 Sobolev 空间上的算子理论”的合同及佐证材料.....23

6. 参与：关于国家自然科学数学天元基金“函数空间在 BMO-Teichmüller 理论上的应用”项目的合同及佐证材料..28

三、论文、著作等

1. 检索证明.....33

2. 以第一作者发表本专业论文情况

2.1 The Kobayashi metric and Gromov hyperbolicity on pseudoconvex domains of finite type in C^2.....35

2.2 The Gehring-Hayman type theorem on pseudoconvex

domains of finite type in C^2..... 55
2.3 Carleson measures on convex domains with smooth boundary of finite type.....70
3.以通讯作者发表本专业论文情况
3.1 The theory of Hardy-Sobolev spaces over the upper half-plane 83
3.2 多复变数准凸映射推广的 Fekete–Szegö 不等式及其应用.....89
3.3 Boundedness of multiparameter Forelli-Rudin type operators on product L^p spaces over tubular domains..... 121
3.4 Composition operators on Dirichlet spaces over the half-plane..... 162
3.5 Toeplitz operators on weighted Bergman spaces over tubular domains..... 176
3.6 BMO on weighted Bergman spaces over tubular domain.. 188

四、其他业绩

1. 指导学生学科竞赛
1.1 指导本科生获全国大学生数学竞赛一等奖.....215
2. 增加主持的 A 类科研项目：关于国家自然科学数学天元基金“2024 复分析及其应用专题讲习班”项目的立项通知及佐证材料.....218

广 东 省 教 育 厅

粤教高函〔2024〕30号

广东省教育厅关于公布2024年度广东省本科高校教学质量与教学改革工程建设项目立项名单的通知

各本科高校:

按照《广东省教育厅关于开展2024年度广东省本科高校教学质量与教学改革工程项目申报推荐工作的通知》等文件安排，经学校遴选推荐、省教育厅审核、公示、复审等环节，现将2024年省本科高校质量工程建设项目立项名单予以公布，并就有关事项通知如下:

一、立项情况

确定立项建设省级实验教学示范中心35个、教学研究中心1个、校企联合实验室31个、科产教融合实践教学基地66个、大学生社会实践教学基地47个、教师教学发展中心3个、课程教研室147个、现代产业学院32个、专项人才培养计划54项、高等教育教学改革项目850项。具体立项名单见附件。

二、项目管理

（一）本次公布项目均为省质量工程建设项目，建设项目经学校组织建设、校内结题并通过省教育厅统一组织项目验收后，正式认定为省级项目。

（二）项目正式实施前，请确保已对项目建设目标、建设举措、预期成果、建设进度安排等进行科学论证，论证专家应不少于5人，且至少有三分之一来自外校。论证后的目标、任务等将作为项目结题验收时的重要依据。

（三）项目日常管理由学校主管部门负责，学校应统筹做好项目中期检查、校内结题验收等工作。校内结题时，邀请校外评审专家人数不得少于专家总人数的三分之二。满足以下条件的项目，经学校正式申请，可以参与省教育厅统一组织的项目验收:

1.项目已完成立项时设定的主要建设任务和目标;

2.项目已取得标志性建设成果，且该成果已在教学实践中得到检验和有效应用;

3.已按照要求完成项目校内结题;

4.符合当年度省统一验收规定的其他条件。

各校质量工程建设项目管理情况，将作为学校下一年度项目立项限额的参考依据。

（四）项目实施过程中，其名称、建设内容（任务）、建设目标、建设周期、主要负责人、预期成果等发生重大变更的，需由时任项目负责人在发生变更后及时提出，经学校项目主管

部门审核后由学校正式来函说明原因；擅自变更上述内容的，验收评定时列为不通过。

三、其他事项

（一）2024 年度各校向省教育厅推荐并获得立项的项目，学校须将项目校内评审、推荐及论证相关材料妥善保存，留底备查。

（二）各校要统筹本校“冲补强”提升计划资金及自有资金对立项项目予以资助，项目获得学校资助情况将作为项目结题验收时重要考察因素之一。如项目建设中取得具有推广价值的优秀成果，请及时形成书面材料报省教育厅高教处。

联系人：李成军，联系电话：020-37626882。

附件：2024 年度广东省本科高校教学质量与教学改革工程建设项目立项名单

广东省教育厅

2024 年 12 月 19 日

附件：

2024年度广东省本科高校教学质量与教学改革工程拟立项项目名单

序号	项目类型	单位名称	项目名称	项目负责人
1	实验教学示范中心	中山大学	中山大学医学院基础医学实验教学中心	田国宝
2	实验教学示范中心	中山大学	微电子科学与工程实验教学中心	虞志益
3	实验教学示范中心	华南理工大学	生物医学创研创新实验中心	付晓玲
4	实验教学示范中心	华南理工大学	集成电路教学与创新实验中心	李斌
5	实验教学示范中心	华南理工大学	电工电子实验教学示范中心	汪娟娟
6	实验教学示范中心	暨南大学	密码科学与技术实验教学示范中心	赖俊祚
7	实验教学示范中心	南方医科大学	法医病理学实验教学示范中心	乔东访
8	实验教学示范中心	南方医科大学	医学人才临床综合能力培养数字化实验教学示范中心	沈洁
9	实验教学示范中心	广州中医药大学	中医养生学实验教学示范中心	黎晖
10	实验教学示范中心	广州中医药大学	临床技能培训实验教学示范中心	邬素珍
11	实验教学示范中心	华南师范大学	教育学实验教学示范中心	李盛兵
12	实验教学示范中心	广东工业大学	人因环境实验教学示范中心	周祥
13	实验教学示范中心	广东海洋大学	数字经济与产业发展综合实验教学示范中心	周昌仕
14	实验教学示范中心	广东药科大学	中医临床技能教学示范中心	肖炜
15	实验教学示范中心	岭南师范学院	地理学实验教学示范中心	牛东风
16	实验教学示范中心	广州大学	心理学实验教学示范中心	聂衍刚
17	实验教学示范中心	广州医科大学	医学模拟教育中心	刘翔
18	实验教学示范中心	深圳大学	前沿交叉学科教学实验中心	李猛
19	实验教学示范中心	南方科技大学	南科大人工智能实验教学示范中心	郝祁
20	实验教学示范中心	惠州学院	材料测试与分析实验教学示范中心	封科军
21	实验教学示范中心	惠州学院	大学物理实验教学示范中心	龙达峰
22	实验教学示范中心	东莞理工学院	机械工程学院实验教学示范中心	郭建文
23	实验教学示范中心	佛山大学	资源环境实验教学中心	张永利
24	实验教学示范中心	广东白云学院	智能制造技术实验教学示范中心	柳贵东
25	实验教学示范中心	广东科技学院	智能制造技术实验教学示范中心	严其艳
26	实验教学示范中心	广东东软学院	计算机系统原理实验教学中心	周富肯
27	实验教学示范中心	广东理工学院	机械工程实验教学中心	杨斌
28	实验教学示范中心	珠海科技学院	工业机器人实验教学示范中心	张恩光
29	实验教学示范中心	广东外语外贸大学南国商学院	计算机与人工智能实验教学示范中心	朱子江
30	实验教学示范中心	广州理工学院	人工智能实验教学示范中心	王煜林
31	实验教学示范中心	广州软件学院	智能电子与控制实验教学中心	甘俊英
32	实验教学示范中心	北京师范大学珠海校区	北京师范大学珠海校区-地理实验教学示范中心	谢云
33	实验教学示范中心	肇庆医学院	基础医学实验教学示范中心	郑恒
34	实验教学示范中心	深圳技术大学	电路与电子智慧化实验教学示范中心	张文伟
35	校企联合实验室	华南理工大学	智能计算联合实验室	何克晶
36	校企联合实验室	暨南大学	暨南大学-豪密科技校企联合实验室	黄书强
37	校企联合实验室	暨南大学	暨南大学包装- 红塔仁恒校企联合实验室	林勤保
38	校企联合实验室	华南农业大学	华南农业大学-汕头保税区联通工业环境友好高分子材料联合实验室	张超群
39	校企联合实验室	南方医科大学	高校-血站-企业细胞工程联合实验室	李婷婷
40	校企联合实验室	南方医科大学	智能医学成像与诊断技术校企联合实验室	路利军
41	校企联合实验室	南方医科大学	法医生物物证精准鉴识联合实验室	朱波峰
42	校企联合实验室	华南师范大学	环境工程设计与模拟仿真联合实验室	张刚
43	校企联合实验室	广东工业大学	机器人及智能制造技术校企联合实验室	陈辞
44	校企联合实验室	广东外语外贸大学	基于大模型的智能会计综合实验室	左志刚
45	校企联合实验室	广东财经大学	金融区块链实验室	胡凡
46	校企联合实验室	广东医科大学	广东医科大学一兰卫医学实验技术校企联合实验室	张晶晶
47	校企联合实验室	广东海洋大学	力学分析校企联合实验室	陈扬枝
48	校企联合实验室	广东海洋大学	农业机械智能化技术与应用实验室	侯明鑫
49	校企联合实验室	广东药科大学	化妆品产教融合校企联合实验室	曹华
50	校企联合实验室	星海音乐学院	全息数智化校企联合实验室	李伟

511	高等教育教学改革项目	华南农业大学	新农科引领下林学一流专业质量提升关键路径探索与实践	刘效东
512	高等教育教学改革项目	华南农业大学	新农科背景下《研究方法与论文写作》的教学改革与探索	苗建银
513	高等教育教学改革项目	华南农业大学	“新农科”背景下科教融汇提升学生创新能力的基因工程实验改革探索与实践	周海
514	高等教育教学改革项目	华南农业大学	《动物营养学》课程思政建设与教学模式创新	谭成全
515	高等教育教学改革项目	华南农业大学	人工智能在数学教学辅助中的应用与探索 —以数学专业核心课程《实变函数与泛函分析》为例	李海绸
516	高等教育教学改革项目	华南农业大学	产教融合背景下多元协同培育创新型人才模式探索与实践——以管理科学类专业为例	韦婷婷
517	高等教育教学改革项目	华南农业大学	思政教育提升《物理化学》理论与实验课程质量的创新实践策略	杨思源
518	高等教育教学改革项目	华南农业大学	基于虚拟仿真技术的环境工程专业“新工科沉浸式实践教学”模式探索	梁瑜海
519	高等教育教学改革项目	华南农业大学	基于实践创新能力培养的《养猪学》课程教学改革研究	洪林君
520	高等教育教学改革项目	华南农业大学	新工科背景下一流农林院校《材料物理》课程教学改革与实践	张学杰
521	高等教育教学改革项目	华南农业大学	数智时代下基于社会情绪和创新能力提升的课程教学改革—组织行为学为例	徐峰
522	高等教育教学改革项目	华南农业大学	新工科背景下风景园林设计表达课程教学改革研究	刘京一
523	高等教育教学改革项目	华南农业大学	基于新时代就业形势的生涯教育本土化探索与课程改革	卢小圣
524	高等教育教学改革项目	华南农业大学	数字人文视域下思政元素融入农林院校美育课程的创新实践研究	李春阳
525	高等教育教学改革项目	华南农业大学	国际化背景下基于“双一流”建设的作物学本科生全英课程教学模式研究—以《生物信息学》为例	张慧
526	高等教育教学改革项目	华南农业大学	“双碳”目标下《生物质复合材料加工方法》课程思政教学改革与实践	郝笑龙
527	高等教育教学改革项目	南方医科大学	以本为本 成果导向 质量为先 全员激励的本科人才培养绩效评价体系构建与实践	蔡谱
528	高等教育教学改革项目	南方医科大学	基于POA的《医学英语》课程思政改革与实践	蔡思颖
529	高等教育教学改革项目	南方医科大学	知识图谱联合AI技术在《内分泌与代谢》整合课程中的应用	程彦臻
530	高等教育教学改革项目	南方医科大学	《健康大数据管理》混合式教学改革优化与实践	陈江芸
531	高等教育教学改革项目	南方医科大学	评估生成式人工智能对高校教师混合式教学设计能力的影响——一项基于SAMR模型框架的实证研究	陈曦
532	高等教育教学改革项目	南方医科大学	基于“科研-教学-学习连结体”理论的医科院校基础学科拔尖创新人才培养机制研究	陈昕昀
533	高等教育教学改革项目	南方医科大学	本研贯通法医科研人才培养机制研究与实践	陈训财
534	高等教育教学改革项目	南方医科大学	智慧教育赋能医学通识课程群的教学改革与实践	陈瑜
535	高等教育教学改革项目	南方医科大学	基于OBE理念的岗位胜任力导向检验医学创新综合教学活动研究	邸玉玮
536	高等教育教学改革项目	南方医科大学	基于生成式人工智能的探究式教学模式在医学专业课程教学中的应用研究——以"临床诊断学"为例	何栩
537	高等教育教学改革项目	南方医科大学	基于EBM和CBCR的抛锚式翻转课堂在口腔医学本科生临床思维培养中的应用研究	胡琛
538	高等教育教学改革项目	南方医科大学	大湾区国际化医教协同基地的探索与实践	贾志敏
539	高等教育教学改革项目	南方医科大学	《病理学》课程知识图谱建设与实践	梁莉

关于公示2024年度校级本科教学质量与教学改革工程项目拟立项名单的通知

来源单位及审核人： 编辑： 审核发布：本科生院（招生办公室） 发布时间：2024-07-29

各学院、部处、各单位：

根据《关于开展2024年度校级本科教学质量与教学改革工程项目申报工作的通知》精神,经项目负责人申报、所在单位推荐和学校组织专家评审等程序，拟立项“基于大湾区新能源汽车人才需求的智能网联技术课程改革”等121个项目为2024年度校级教学改革项目；拟立项“涉外法治人才培养实验班”等36个项目为2024年度校级质量工程项目；根据学校年度人才培养工作重点及本年度申报项目质量，拟立项“‘长基计划’下新文科历史学课程体系的改革与实践”等11个项目为2024年度“长基计划”教学质量和教学改革工程专项项目（经费由“长基计划”专业专项资金资助）。教改招标项目因不符合立项指南要求，不予立项。现予以公示，具体名单见附件。

公示期自2024年7月29日至8月3日。如有异议，请在公示期内以书面方式提交（附必要的证据材料，并署真实姓名），未署真实姓名或逾期者不予受理。

附件：1.华南农业大学2024年校级教学改革拟立项名单.xlsx

2.华南农业大学2024年校级质量工程拟立项名单.xlsx

3.华南农业大学2024年度校级“长基计划”教学质量和教学改革工程专项拟立项名单.xlsx

联系人：孙齐胜
电 话：85288020（15192666549）
地 址：行政楼329
邮 箱：gjyj@scau.edu.cn

最新动态

2026-04-07
华南农业大学关于广东省高校军
程拟推荐名单的公示

2026-04-02
关于开展新增通识教育选修课程
知

2026-03-30
关于开展校级课程思政示范项目
题验收工作的通知

2026-03-27
关于做好广东省本科教学质量与
项目2025年度验收工作...

2026-03-16
关于2026年上半年全国大学英语
试报名的通知

附件3

华南农业大学2024年“长基计划”校级教学质量和教学改革工程专项拟立项名单

序号	项目类型	项目类别	单位	项目名称	负责人	职称	项目组成员	备注
1	教学改革项目	重点项目	人文与法学学院	“长基计划”下新文科历史学课程体系的改革与实践	衷海燕	教授	赵艳萍、赵飞、殷小平、龚金红	长基计划（历史学）
2		重点项目	生命科学学院	“新农科”背景下学研融合提升学生创新能力的基因工程实验改革探索与实践	周海	研究员	易继财、郑少燕、姜大刚	长基计划（生物科学）
3		重点项目	生命科学学院	优化课程体系、提高教学质量——以“生物化学实验”课程为例	陈庆梅	讲师	母培强、许燕珍、吴骏、林如琴	长基计划（生物科学）
4		重点项目	数学与信息学院、软件学院	人工智能在数学教学辅助中的应用与探索—	李海纲	副教授	房少梅、雷春林、李朗、王雪琴	长基计划（数学与应用数学）
5		自筹项目	资源环境学院	农业院校全英专业课的课程思政体系建设探索——以“普通生态学”为例	谈凤笑	副教授	舒迎花、陈桂葵、蔡一霞	长基计划（生态学）
6		自筹项目	资源环境学院	“案例教学+探究式研讨+网络教学”在《全球变化》中的实践探索	余玲玲	助理研究员	危晖、李荣华	长基计划（生态学）
1	质量工程项目	课程教研室	资源环境学院	普通生态学课程教研室	舒迎花	教授	王建武、陈桂葵、危晖、李荣华、谈凤笑、陈永健	长基计划（生态学）
2		课程教研室	数学与信息学院、软件学院	农科数学课程教研室	杨德贵	教授	肖莉、李凤、岑冠军、杨志程、周燕、陆琪、何婧、夏英俊、丁仕虹、刘景荣、储格斌	长基计划（数学与应用数学）
3		课程教研室	人文与法学学院	农业历史教研室	赵艳萍	副教授	王福昌、赵飞、陈志国、龚金红、金格格、衷海燕、刘玲娣	长基计划（历史学）
4		科产教融合实践教学基地	数学与信息学院、软件学院	华南农业大学广州维德科技有限公司科产教融合实践教学基地	张昕	副教授	袁利国、陈羽、方平、邱华、周裕中、陈银辉	长基计划（数学与应用数学）
5		大学生社会实践教学基地	生命科学学院	广东潮安凤凰山省级自然保护区植物学野外实习基地	张荣京	副教授	郝刚、耿世磊、龚维、孔德鑫、白玫、谢建光	长基计划（生物科学）

国家自然科学基金资助项目批准通知

（预算制项目）

李海绸　先生/女士：

根据《国家自然科学基金条例》、相关项目管理办法规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12326407，项目名称：2024复分析及其应用专题讲习班 ，直接费用：20.00万元，项目起止年月：2024年01月至 2024年 12月，有关项目的评审意见及修改意见附后。

请您尽快登录科学基金网络信息系统（https://grants.nsfc.gov.cn），**认真阅读《国家自然科学基金资助项目计划书填报说明》并按要求填写《国家自然科学基金资助项目计划书》（以下简称计划书）**。对于有修改意见的项目，请您按修改意见及时调整计划书相关内容；如您对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

请您将电子版计划书通过科学基金网络信息系统（https://grants.nsfc.gov.cn）提交，由依托单位审核后提交至自然科学基金委。自然科学基金委审核未通过者，将退回的电子版计划书修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印）并在项目负责人承诺栏签字，由依托单位科研、财务管理等部门审核、签章并在承诺栏加盖依托单位公章，且将申请书纸质签字盖章页订在其中一份计划书之后，一并报送至自然科学基金委项目材料接收工作组。纸质版计划书应当保证与审核通过的电子版计划书内容一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委提交电子版计划书、报送纸质版计划书并补交申请书纸质签字盖章页截止时间节点如下：

1. **2024年01月22日16点：**提交电子版计划书的截止时间；

2. **2024年01月29日16点：**提交修改后电子版计划书的截止时间；

3. **2024年02月05日：**报送纸质版计划书（一式两份，其中一份包含申请书纸质签字盖章页）的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，逾期不报计划书或申请书纸质签字盖章页且未说明理由的，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2024年01月02日

国家自然科学基金

资助项目准予结题通知

李海绸　同志：

您承担的国家自然科学基金项目：(2024复分析及其应用专题讲习班)，批准号：(12326407)按有关规定已审核完毕，准予结题。

与本项目资助有关的后续成果，请您继续及时报送。

祝您在研究工作中取得更好的成绩！

国家自然科学基金委员会

数理科学部

2025年3月31日

国家自然科学基金资助项目批准通知

（预算制项目）

李海绸　先生/女士：

根据《国家自然科学基金条例》、相关项目管理办法规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12226318，项目名称：函数空间在BMO-Teichmüller理论上的应用，直接费用：10.00万元，项目起止年月：2023年01月至2023年12月，有关项目的评审意见及修改意见附后。

请您尽快登录科学基金网络信息系统（https://grants.nsfc.gov.cn），**认真阅读《国家自然科学基金资助项目计划书填报说明》并按要求填写《国家自然科学基金资助项目计划书》（以下简称计划书）**。对于有修改意见的项目，请您按修改意见及时调整计划书相关内容；如您对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

请您将电子版计划书通过科学基金网络信息系统（https://grants.nsfc.gov.cn）提交，由依托单位审核后提交至自然科学基金委。自然科学基金委审核未通过者，将退回的电子版计划书修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印）并在项目负责人承诺栏签字，由依托单位科研、财务管理等部门审核、签章并在承诺栏加盖依托单位公章，且将申请书纸质签字盖章页订在其中一份计划书之后，一并报送至自然科学基金委项目材料接收工作组。纸质版计划书应当保证与审核通过的电子版计划书内容一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委提交电子版计划书、报送纸质版计划书并补交申请书纸质签字盖章页截止时间节点如下：

1. **2022年10月28日16点：**提交电子版计划书的截止时间；

2. **2022年11月04日16点：**提交修改后电子版计划书的截止时间；

3. **2022年11月11日：**报送纸质版计划书（一式两份，其中一份包含申请书纸质签字盖章页）的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，逾期不报计划书或申请书纸质签字盖章页且未说明理由的，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2022年10月08日

国家自然科学基金

资助项目准予结题通知

李海绸　同志：

您承担的国家自然科学基金项目：(函数空间在BMO-Teichmüller理论上的应用)，批准号：(12226318)按有关规定已审核完毕，准予结题。

与本项目资助有关的后续成果，请您继续及时报送。

祝您在研究工作中取得更好的成绩！

国家自然科学基金委员会

数理科学部

2024年4月16日

受理编号：c25140500002019

项目编号：2025A1515011213

文件编号：粤基金字〔2025〕10号

广东省基础与应用基础研究基金项目

任务书

2025A1515011213

项目名称： 拟凸域上的解析函数空间及其算子理论研究

项目类别： 广东省自然科学基金-面上项目

项目起止时间： 2025-01-01　至　2027-12-31

管理单位（甲方）： 广东省基础与应用基础研究基金委员会

依托单位（乙方）： 华南农业大学

通讯地址： 广东省广州市天河区五山路483号

邮政编码： 510642　　**单位电话：** 020-85283435

项目负责人： 李海绸　　**联系电话：** 020-85285393

（广东科技微信公众号）　　（查看任务书信息）　　（受理纸质材料二维码）

广东省基础与应用基础研究
基金委员会
二〇二〇年制

八、本任务书签约各方

管理单位（甲方）：　　广东省基础与应用基础研究基金委员会　（盖章）

法定代表人（或法人代理）：　曾路　（签章）

2025 年　03 月　21 日

依托单位（乙方）：　华南农业大学　（盖章）

法定代表人（或法人代理）：　薛红卫　（签章）

联系人（项目主管）姓名：　夏杰　（签章）

Email：　kjcgxk@scau.edu.cn

电话：　020-85283435　/　13711345768

开户单位名称：　华南农业大学

开户银行名称：　广东广州工行五山支行

开户银行账号：　3602002609000310520

2025年4月9日

联系人（项目负责人）姓名：　李海绸　（签名）

Email：　hcl2016@scau.edu.cn

电话：　020-85285393

2025 年 4 月 8 日

广东省科学技术厅文件

粤科农字〔2024〕200号

广东省科学技术厅关于下达“百千万工程”农村科技特派员第二轮重点派驻人员名单和任务清单的通知

各有关地市科技局，各农村科技特派员派出单位:

为深入贯彻落实省委关于实施“百县千镇万村高质量发展工程”有关决策部署，进一步推动农村科技特派员下乡助力城乡区域协调发展，按照《广东省农村科技特派员科技助力百县千镇万村高质量发展行动计划（2023-2026）》《广东省科学技术厅关于落实“百千万工程”开展新一轮农村科技特派员重点派驻任务（2024-2026年）选派管理工作的通知》要求，在第一轮农村科技特派员3年帮扶重点派驻任务即将到期的情况下，按照“镇村

发榜+地市组织选派+省级认定支持”的新型组织管理方式，第二轮共选派2960名农村科技特派员组建984个团队，“一对一”结对科技帮扶全省涉农县（市、区）辖内984个乡镇。现将“百千万工程”农村科技特派员第二轮重点派驻人员名单和任务清单印发给你们:

一、各地市科技管理部门、派出单位应按照《广东省科学技术厅关于落实“百千万工程”开展新一轮农村科技特派员重点派驻任务（2024-2026年）选派管理工作的通知》（粤科函农字〔2023〕1649号）、《广东省科学技术厅关于印发〈农村科技特派员助力“百千万工程”管理工作指引（2024-2026年）〉的通知》（粤科函农字〔2024〕192号）有关要求，做好本辖区、本单位农村科技特派员组织、管理、考核和验收相关工作。

二、各农村科技特派员团队应聚焦落实省委实施“百千万工程”决策部署，在省、市科技管理部门指导下，扎根基层一线，配合驻镇帮扶工作队开展科技帮扶工作，农村科技特派员团队纳入驻镇帮扶工作队（乡镇人民政府）统一管理，本轮帮扶任务周期为3年。

三、农村科技特派员承担重点派驻任务经费保障，按年度以科技支撑“百千万工程”专项资金下达到各地市，各地市科技管理部门要加强管理，压实责任，专款专用，提高资金使用效率，应将专项资金拨付到相关派出单位，用于开展科技帮扶工作。

附件：“百千万工程”农村科技特派员第二轮重点派驻人员名单和任务清单

省科技厅

2024年9月5日

（联系人及电话：赵清泉，020-83163909）

附件

“百千万工程”农村科技特派员第二轮重点派驻人员名单和任务清单

序号	编号	帮扶镇（乡）	派出单位	团队成员	任务名称
一、汕头市					
1	KTP20240001	潮南区两英镇	汕头市乡村振兴发展中心	陈梓敏、杨伟权、郑秋枫	麻叶的种植与加工技术研究及示范推广
2	KTP20240002	潮南区司马浦镇	广东省农业科学院（负责人、成员2）、汕头市农业科学研究所（成员1）	董博、张建林、黄丹	农田综合利用与乡镇景观关键技术示范推广
3	KTP20240003	潮南区陈店镇	广东开放大学（负责人、成员2）、汕头职业技术学院（成员1）	王紫虹、刘毅嘉、罗学强	乡镇特色产业电商综合服务示范与推广
4	KTP20240004	潮南区胪岗镇	汕头市农业科学研究所	汤昌兴、张秀珊、王玉	粮油花卉等作物新品种及栽培技术研究与示范推广
5	KTP20240005	潮南区仙城镇	仲恺农业工程学院（负责人）、汕头市农业科学研究所（成员1、成员2）	陈青春、王兴才、王文举	鲜食玉米、水稻等作物高效周年轮作模式推广
6	KTP20240006	潮南区成田镇	广东省农业科学院（负责人、成员2）、汕头市农业科学研究所（成员1）	闫晋强、郭英铎、蔡金森	“成甜”牌水果玉米周年生产技术支撑
7	KTP20240007	潮南区陇田镇	仲恺农业工程学院（负责人、成员2）、汕头市白沙蔬菜原种研究所（成员1）	樊丽娜、郭商羿、齐永文	甘薯、玉米产业提质增效技术示范推广

496	KTP20240496	雷州市唐家镇	华南农业大学（负责人、成员 1）、广东海洋大学（成员 2）	康云艳、黄日明、王延芹	优质蔬菜新品种引进及产业化示范与推广
497	KTP20240497	雷州市雷高镇	中国水产科学研究院南海水产研究所	李俊伟、张书飞、饶义勇	红树林种植与水产养殖耦合技术示范与培训推广
498	KTP20240498	雷州市覃斗镇	广东海洋大学	邓岳文、杨创业、廖永山	海水珍珠贝优质苗种推广
499	KTP20240499	雷州市东里镇	广东药科大学	蔡延渠、袁旭江、刘文彬	优质南美白对虾的中草药替抗健康养殖技术推广与应用
500	KTP20240500	雷州市调风镇	仲恺农业工程学院	方凯、刘洁、孙昊颖	调风镇特色产业培育
501	KTP20240501	雷州市松竹镇	仲恺农业工程学院	郭鹏飞、张垒、张超龙	松竹镇水稻生产及农产品推广信息服务平台构建
502	KTP20240502	雷州市南兴镇	广东省农业科学院（负责人、成员 1）、湛江市农业科学研究院（成员 2）	张彬、杨梯丰、林建强	南兴镇稻田多熟高效种植模式构建与示范应用
503	KTP20240503	雷州市白沙镇	华南农业大学	李海绸、吴道铭、李湛波	智慧农业科技特派团
504	KTP20240504	雷州市附城镇	广东科贸职业学院	马力、李春燕、陈锦泓	雷州市附城镇乡村全领域服务支撑项目
505	KTP20240505	雷州市英利镇	广东技术师范大学（负责人）、广东科贸职业学院（成员 1）、广东东软学院（成员 2）	陈贞丰、潘冰、潘晓	科技赋能助力农作物提质增产
506	KTP20240506	雷州市龙门镇	广东科贸职业学院	高广宇、陈瑞克、蔡肖肖	雷州市龙门镇农村产业一体化服务支撑项目
507	KTP20240507	雷州市客路镇	广东科贸职业学院	陈振卓、刘家辉、晏辰洁	雷州市客路镇雷琼黄牛产业技术支撑项目
508	KTP20240508	雷州市杨家镇	广东海洋大学（负责人）、湛江市农业科学研究院（成员 1、成员 2）	陈苗、余小丽、蔡伟俊	番薯健康种苗生产与推广
509	KTP20240509	雷州市纪家镇	中国热带农业科学院农业机械研究所	周思理、王系林、代叶	薯类、菠萝等作物机械化技术示范推广

国家自然科学基金资助项目批准通知

曹广福　先生/女士：

根据《国家自然科学基金条例》规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12071155，项目名称：解析Sobolev空间上的算子理论 ，直接费用：51.00万元，项目起止年月：2021年01月至 2024年 12月，有关项目的评审意见及修改意见附后。

请尽早登录科学基金网络信息系统（https://isisn.nsfc.gov.cn），获取《国家自然科学基金资助项目计划书》（以下简称计划书）并按要求填写。对于有修改意见的项目，请按修改意见及时调整计划书相关内容；如对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

电子版计划书通过科学基金网络信息系统（https://isisn.nsfc.gov.cn）上传，依托单位审核后提交至自然科学基金委进行审核。审核未通过者，返回修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印），依托单位审核并加盖单位公章，将申请书纸质签字盖章页订在其中一份计划书之后，一并将上述材料报送至自然科学基金委项目材料接收工作组。电子版和纸质版计划书内容应当保证一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委补交申请书纸质签字盖章页、提交和报送计划书截止时间节点如下：

1. **2020年10月14日16点：**提交电子版计划书的截止时间（视为计划书正式提交时间）；

2. **2020年10月21日16点：**提交电子修改版计划书的截止时间；

3. **2020年10月28日16点：**报送纸质版计划书（其中一份包含申请书纸质签字盖章页）的截止时间。

4. **2020年11月18日16点：**报送修改后的申请书纸质签字盖章页的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，未说明理由且逾期不报计划书或申请书纸质签字盖章页者，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2020年9月18日

项目批准号	12071155
申请代码	A010602
归口管理部门	
依托单位代码	51064208A0499-0932

国家自然科学基金委员会

资助项目计划书

资助类别：面上项目

亚类说明：

附注说明：

项目名称：解析Sobolev空间上的算子理论

直接费用：51万元　　执行年限：2021.01-2024.12

负 责 人：曹广福

通讯地址：广州市天河区五山华南农业大学

邮政编码：510642　　电　　话：85280070

电子邮件：guangfucao@163.com

依托单位：华南农业大学

联 系 人：倪慧群　　电　　话：020-85280070

填表日期：2020年09月18日

国家自然科学基金委员会制

Version：1.005.131

简表

项目负责人信息	姓　　名	曹广福	性 别	男	出生年月	1960年11月	民 族	汉族
	学　　位	博士			职称	教授		
	是否在站博士后	否		电子邮件		guangfucao@163.com		
	电　　话	85280070		个人网页				
	工 作 单 位	华南农业大学						
	所 在 院 系 所	数学与信息(软件)学院						
依托单位信息	名　　称	华南农业大学					代码	51064208A0499
	联 系 人	倪慧群		电子邮件		kycjhk@scau.edu.cn		
	电　　话	020-85280070		网站地址		http://kjc.scau.edu.cn/		
合作单位信息	单 位 名 称							
	广州大学							
项目基本信息	项 目 名 称	解析Sobolev空间上的算子理论						
	资 助 类 别	面上项目			亚 类 说 明			
	附 注 说 明							
	申 请 代 码	A010602:算子理论与算子代数						
	基 地 类 别							
	执 行 年 限	2021.01-2024.12						
	直 接 费 用	51万元						

项目组主要成员

编号	姓名	出生年月	性别	职称	学位	单位名称	电话	证件号码	项目分工	每年工作时间（月）
1	曹广福	1960.11	男	教授	博士	华南农业大学	85280070	220104196011121510	项目负责人	9
2	李海绸	1985.10	女	副教授	博士	华南农业大学	13688868352	440882198510177225	承担复合算子的研究	9
3	刘丹	1978.08	女	副教授	博士	华南农业大学	13570384468	420114197808180029	承担Toeplitz算子的研究	9
4	秦杰	1992.12	男	博士生	硕士	广州大学	18623256267	500230199212244372	参与	9
5	张艺渊	1994.04	女	博士生	硕士	广州大学	18819488097	441826199404181727	参与	9
6	李金燕	1995.10	女	硕士生	学士	华南农业大学	18819260500	440982199510151500	参与	9
7	郑佳鸿	1994.11	女	硕士生	学士	华南农业大学	15767976584	445122199411014740	参与	9

总人数	高级	中级	初级	博士后	博士生	硕士生
7	3	0	0	0	2	2

项目批准号	12226334
申请代码	A0201
归口管理部门	
依托单位代码	10019008B1321-2425

国家自然科学基金

资助项目计划书

（预算制项目）

资助类别：数学天元基金项目

亚类说明：数学天元基金

附注说明：天元数学访问学者项目

项目名称：函数空间在BMO-Teichmüller理论上的应用

直接费用：20万元　　执行年限：2023.01-2023.12

负 责 人：刘劲松

通讯地址：中国科学院数学与系统科学研究院数学所

邮政编码：100190　　电　　话：010－82541821

电子邮件：liujsong@math.ac.cn

依托单位：中国科学院数学与系统科学研究院

联 系 人：王晓欢　　电　　话：82541770

填表日期：2022年10月15日

国家自然科学基金委员会制

Version：1.004.530

简表

项目负责人信息	姓　　名	刘劲松	性 别	男	出生年月	1975年01月	民 族	汉族
	学　　位	博士			职称	研究员		
	是否在站博士后	否			电子邮件	liujsong@math.ac.cn		
	电　　话	010－82541821			个人网页			
	工 作 单 位	中国科学院数学与系统科学研究院						
	所 在 院 系 所							
依托单位信息	名　　称	中国科学院数学与系统科学研究院					代码	10019008B1321
	联 系 人	王晓欢			电子邮件	xhwang@amss.ac.cn		
	电　　话	82541770			网站地址	http://www.amss.ac.cn		
合作单位信息	单 位 名 称							
	华南农业大学							
项目基本信息	项 目 名 称	函数空间在BMO-Teichmüller理论上的应用						
	资 助 类 别	数学天元基金项目			亚 类 说 明	数学天元基金		
	附 注 说 明	天元数学访问学者项目						
	申 请 代 码	A0201:单复变函数论						
	基 地 类 别							
	执 行 年 限	2023.01-2023.12						
	直 接 费 用	20万元						

项目组主要成员

编号	姓名	出生年月	性别	职称	学位	单位名称	电话	证件号码	项目分工	每年工作时间（月）
1	刘劲松	1975.01	男	研究员	博士	中国科学院数学与系统科学研究院	010－82541821	432522197501200350	项目负责人	10
2	李海绸	1985.10	女	副教授	博士	华南农业大学	020-85280322	440882198510177225	项目参与人	10

总人数	高级	中级	初级	博士后	博士生	硕士生
2	2	0	0	0	0	0

国家自然科学基金项目负责人、依托单位承诺书

国家自然科学基金项目负责人承诺书

本人郑重承诺：我接受国家自然科学基金的资助，严格遵守中共中央办公厅、国务院办公厅《关于进一步加强科研诚信建设的若干意见》《关于进一步弘扬科学家精神加强作风和学风建设的意见》《关于加强科技伦理治理的意见》等规定，及国家自然科学基金委员会关于资助项目管理、项目资金管理等各项规章，在《计划书》填写及项目执行过程中：

（一）按照《批准通知》《国家自然科学基金资助项目计划书填报说明》的要求填写《计划书》，未自行降低、更改目标任务或约定要求，或缩减研究（研制）内容；

（二）树立“红线”意识，严格履行科研合同义务，按照《计划书》负责实施本项目（批准号：12226334），切实保证研究工作时间，按时报送有关材料，及时报告重大情况变动，不违规将科研任务转包、分包他人，不以项目实施周期外或不相关成果充抵交差；

（三）遵守科研诚信、科技伦理规范和学术道德，认真开展研究工作，对资助项目发表的论著和取得的研究成果按规定进行标注，不在非本项目资助的成果或其他无关成果上标注本项目批准号，反对无实质学术贡献者“挂名”，不在成果署名、知识产权归属等方面侵占他人合法权益，并如实报告本人及项目组成员发生的违背科研诚信要求的任何行为；

（四）尊重科研规律，弘扬科学家精神，严谨求实，追求卓越，反对浮夸浮躁、投机取巧，不人为夸大学术或技术价值，不传播未经科学验证的现象和观点；

（五）将项目资金全部用于与本项目研究工作相关的支出，并结合科研活动需要，科学合理安排项目资金支出进度；

（六）做好项目组成员的教育和管理，确保遵守以上相关要求。

如违背上述承诺，本人愿接受国家自然科学基金委员会和相关部门做出的各项处理决定。

项目负责人（签字）：

2022 年 10 月 26 日

依托单位科研管理部门：	依托单位财务管理部门：
负责人（签章）：宋瑞昆 2022 年 10 月 26 日	负责人（签章）：王碧琳 2022 年 10 月 26 日

国家自然科学基金项目依托单位承诺书

我单位同意承担上述国家自然科学基金项目，将保证项目负责人及其研究队伍的稳定和研究项目实施所需的条件，严格遵守国家自然科学基金委员会有关资助项目管理、项目资金管理、科研诚信管理和科技伦理管理等各项规定，并督促实施。

依托单位（公章）

2022 年 10 月 26 日

国家自然科学基金资助项目签批审核表

本栏目由自然科学基金委填写

科学处审查意见：

同意按计划执行

负责人（签章）：

年　月　日

2022年 11月 21日

科学部审查意见：

同意按计划执行

负责人（签章）：

年　月　日

2022年 11月 21日

SCAULIB202626155

检索证明

根据委托人提供的论文材料，委托人华南农业大学数学与信息学院 李海绸(学科类型:自然科学) 9 篇论文收录情况如下表。

序号	论文名称	发表刊物及发表的年月卷期/页码等	作者排名	论文等级	作者文中单位	收录情况	影响因子	中科院大类分区
1	Boundedness of Multiparameter Forelli-Rudin Type Operators on Product L^p Spaces over Tubular Domains	ADVANCES IN APPLIED CLIFFORD ALGEBRAS 出版年：2025 出版日期：JUN 25 卷期：35 3 页码：- 文献号：35 文献类型：Article	共同通讯作者（倒数第一）	B 类	华南农业大学数学与信息学院	SCI	IF2-year=1.2 IF5-year=1.1 （2024）	数学 4 区 Top 期刊：否 OA 期刊：否 （2025）
2	BMO on Weighted Bergman Spaces Over Tubular Domains	COMPLEX ANALYSIS AND OPERATOR THEORY 出版年：2025 出版日期：FEB 27 卷期：19 3 页码：- 文献号：46 文献类型：Article	通讯作者	B 类	华南农业大学数学与信息学院	SCI	IF2-year=0.8 IF5-year=0.9 （2024）	数学 4 区 Top 期刊：否 OA 期刊：否 （2025）
3	Toeplitz operators on weighted Bergman spaces over tubular domains	GEORGIAN MATHEMATICAL JOURNAL 出版年：2025 出版日期：AUG 1 卷期：32 4 页码：621-632 文献类型：Article	通讯作者	B 类	华南农业大学数学与信息学院	SCI	IF2-year=0.7 IF5-year=0.6 （2024）	数学 4 区 Top 期刊：否 OA 期刊：否 （2025）

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2026-03-13

The Journal of Geometric Analysis (2024) 34:10
https://doi.org/10.1007/s12220-023-01450-3

The Kobayashi Metric and Gromov Hyperbolicity on Pseudoconvex Domains of Finite Type in $\mathbb{C}^2$

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Received: 21 June 2023 / Accepted: 20 September 2023 / Published online: 30 October 2023
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Abstract
In this paper, we obtain a more precise estimate of Catlin-type distance for smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. As an application, we get an alternative proof of the Gromov hyperbolicity of this domain equipped with the Kobayashi distance.

Keywords Pseodoconvex domain · Kobayashi metric · Pseudodistance · Gromov hyperbolicity

Mathematics Subject Classification 32F45 · 32T25

1 Introduction

In several complex variables, the Kobayashi metric is an important invariant metric under biholomorphic mappings. And the Gromov hyperbolicity of this metric has been studied in recent years. In [1], Balogh and Bonk firstly obtained a sharp estimate of

H. Li is supported by NSFC (Grant Nos. 12226318, 12226334).

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Kobayashi distance on bounded strongly pseudoconvex domain in $\mathbb{C}^n$. By using the estimate, they obtained the Gromov hyperbolicity of this domain. Later Zimmer [11] proved the Gromov hyperbolicity of convex domain of finite type in $\mathbb{C}^n$ by scaling method. After his work, Fiacchi [5] proved the case of pseudoconvex domain of finite type in $\mathbb{C}^2$.

On the other hand, Catlin [4] and McNeal [8] constructed polydisks on pseudoconvex domain of finite type in $\mathbb{C}^2$ or convex domain of finite type in $\mathbb{C}^n$. Moreover, McNeal also defined a local pseudodistance near a boundary point. Recently, Wang [10] used it to get an estimate of Kobayashi distance on convex domain of finite type in $\mathbb{C}^n$. And it can give a different proof of Zimmer's result in [11].

After Catlin's work, Herbort [6] used the corresponding local pseudodistance to get an estimate of Kobayashi distance on pseudoconvex domain of finite type in $\mathbb{C}^2$. Inspired by Herbort and Wang, we improve the estimate in [7, Theorem 1.12], which can obtain the Gromov hyperbolicity of $(\Omega, d_{\widetilde{K}})$. Here $d_{\widetilde{K}}$ is the Catlin-type distance (see Sect. 3.4). Since Catlin-type metric is comparable to Kobayashi metric, it can give an alternative proof of Theorem 1.1 in [5] which is originally stated as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of finite type with Kobayashi metric d_K, then (Ω, d_K) is a Gromov hyperbolic space.*

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type and U be a neighborhood of $\partial\Omega$. After shrinking U, Herbort [6] defined a pseudodistance $d(\cdot, \cdot)$ in U (see Sect. 3.2). We extend it to Ω which is denoted by $D(\cdot, \cdot)$, and set the function $g : \Omega \times \Omega \to \mathbb{R}$ by

$$g(x, y) = 2 \log \frac{D(x, y) + \delta(x) \vee \delta(y)}{\sqrt{\delta(x)\delta(y)}}$$

where $a \vee b := \max\{a, b\}$.

Note that the construction of g was already present in [1], and similar to [1, Corollary 1.3] we have the following main result.

Theorem 1.2 *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type, then there is a constant $C > 0$ such that*

$$g(x, y) - C \leq d_{\widetilde{K}}(x, y) \leq g(x, y) + C$$

for any $x, y \in \Omega$. Here $d_{\widetilde{K}}$ is the Catlin-type distance.

As an application, we prove the following theorem which implies Theorem 1.1.

Theorem 1.3 *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type, then $(\Omega, d_{\widetilde{K}})$ is a Gromov hyperbolic space.*

From Theorem 1.2, we know that $D(a, b) \asymp \exp(-(a|b)_w^{d_{\widetilde{K}}})$ for $a, b \in \partial\Omega$ with a fixed point $w \in \Omega$. Combining Propositon 2.7, we have the following extension result for quasi-isometry.

Corollary 1.4 *Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}^2$ are smoothly bounded pseudoconvex domains of finite type, and $f : \Omega_1 \to \Omega_2$ is a quasi-isometry with respect to the Kobayashi distances. Then f extends continuously to a map $\bar{f} : \overline{\Omega}_1 \to \overline{\Omega}_2$ such that $\bar{f}|_{\partial\Omega_1} : (\partial\Omega_1, D_1) \to (\partial\Omega_2, D_2)$ is a power quasisymmtery with respect to the pseudodistances. Moreover, it is bi-Hölder continuous with respect to the pseudodistances.*

Remark 1.5 As $|a-b|^m \lesssim D(a,b) \lesssim |a-b|$ for $a, b \in \partial\Omega$, it implies [7, Theorem 1.2] that $\bar{f}|_{\partial\Omega_1}$ is bi-Hölder continuous with respect to the Euclidean distance.

The paper is organized as follows. In Sect. 2 we give the preliminaries will be used. In Sect. 3 we recall the construction of polydisks and Catlin-type metric. In Sect. 4 we give the proofs of Theorems 1.2 and 1.3.

2 Preliminaries

2.1 Notations

(1) We denote $|z|$ as the standard Euclidean norm for $z \in \mathbb{C}^n$ and let $|z_1 - z_2|$ denote the Euclidean distance of $z_1, z_2 \in \mathbb{C}^n$.
(2) Suppose Ω is a bounded domain in $\mathbb{C}^n$, for $x \in \mathbb{C}^n$ we set

$$\delta(x) := \inf\{|x-y| : y \in \partial\Omega\}$$

and denote $N_\epsilon(\partial\Omega) := \{x \in \mathbb{C}^n : \delta(x) < \epsilon\}$ whenever $\epsilon > 0$.
(3) Let $\Delta \subset \mathbb{C}$ be the unit disk and $z, w \in \Delta$. We denote $d_\Delta(z,w)$ as Poincare distance in Δ.
(4) Let f, g be functions, we denote $f \lesssim g$ if there is $C > 0$ such that $f \le Cg$, and we write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.
(5) For $z \in \mathbb{C}^n$, we denote z_i as the i-th component of z.

2.2 Boundary Projection

We give the relationship between a point and its boundary projection. And the proof can be found in [1, Lemma 2.1].

Lemma 2.1 *Suppose $\Omega = \{z \in \mathbb{R}^n : r(z) < 0\}$ is a bounded domain with C^2-smooth boundary. Then there exists a constant $\delta_0 > 0$ such that*

(1) *For every point $x \in N_{\delta_0}(\partial\Omega)$, there is a unique point $\pi(x) \in \partial\Omega$ such that $|x - \pi(x)| = \delta(x)$.*
(2) *The signed distance function*

$$\rho(x) = \begin{cases} -\delta(x) & x \in \Omega \\ \delta(x) & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

is C^2-smooth in $N_{\delta_0}(\partial\Omega)$.

(3) *For the fibers of map* $\pi : N_{\delta_0}(\partial\Omega) \to \partial\Omega$ *we have*

$$\pi^{-1}(p) = (p - \delta_0\vec{n}(p),\ p + \delta_0\vec{n}(p)),$$

where $\vec{n}(p)$ *is the outer unit normal vector of* $\partial\Omega$ *at* $p \in \partial\Omega$.

(4) *The gradient of the defining function* ρ *satisfies*

$$\nabla\rho(x) = \vec{n}(\pi(x)).$$

(5) *The projection* $\pi : N_{\delta_0}(\partial\Omega) \to \partial\Omega$ *is* C^1*-smooth.*

Now we recall that for any $p \in \partial\Omega$, the real tangent space $T_p\partial\Omega$ is given by

$$T_p\partial\Omega = \{X \in \mathbb{C}^n : \mathrm{Re}\langle\bar{\partial}r(p), X\rangle = 0\},$$

and its complex tangent space is

$$H_p\partial\Omega = \{X \in \mathbb{C}^n : \langle\bar{\partial}r(p), X\rangle = 0\},$$

where $\bar{\partial}r(p) = (\dfrac{\partial r}{\partial\bar{z}_1}(p), \ldots, \dfrac{\partial r}{\partial\bar{z}_n}(p))$. Here $\langle\cdot,\cdot\rangle$ is the standard Hermitian product. Therefore, for any $0 \neq X \in \mathbb{C}^n$ it has a unique orthogonal decomposition $X = X_N + X_H$ with $X_H \in H_p\partial\Omega$ and $X_N \in N_p\partial\Omega$. Here $N_p\partial\Omega$ is the complex one-dimensional subspace of $\mathbb{C}^n$ orthogonal to $H_p\partial\Omega$.

2.3 The Invariant Metrics

A $Finsler\ metric$ in Ω is an upper continuous map $F : \Omega \times \mathbb{C}^n \to [0, \infty)$ such that $F(z, tX) = |t|F(z, X)$ for all $z \in \Omega,\ t \in \mathbb{C},\ X \in \mathbb{C}^n$. And the corresponding distance d_F is defined by

$$d_F(x, y) := \inf\{L_F(\gamma) : \gamma : [0, 1] \to \Omega \text{ is a piecewise } C^1\text{-smooth curve with } \gamma(0) = x, \gamma(1) = y\},$$

where $L_F(\gamma) := \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt$.

For a domain $\Omega \subset \mathbb{C}^n$, the Kobayashi metric $K(z, X)$ is defined by

$$K(z, X) := \inf\{|\xi| : \exists f \in \mathrm{Hol}(\Delta, \Omega),\ \text{with}\ f(0) = z, d(f)_0(\xi) = X\}$$

for $z \in \Omega,\ 0 \neq X \in \mathbb{C}^n$, and we denote d_K the corresponding Kobayashi distance. Moreover, the Caratheodory distance is defined by

$$d_C(p, q) := \sup\{d_\Delta(f(p), f(q)) : f : \Omega \to \Delta \text{ isholomorphic}\}.$$

Due to the result in [9, Theorem 1], we know that $d_K(p, q) \geq d_C(p, q)$ for any $p, q \in \Omega$.

2.4 Gromov Hyperbolicity

Now we introduce some definitions and properties about Gromov hyperbolicity. Refer to [3] for details.

Let (X, d) be a metric space. The *Gromov product* of two points $x, y \in X$ with respect to $w \in X$ is defined as follows

$$(x|y)_w := \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).$$

Recall that (X, d) is a *geodesic* metric space if any $x, y \in X$ can be joined by a geodesic segment, i.e., the image of an isometry $\gamma : [0, d(x, y)] \to X$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. And we call a metric space (X, d) is *proper* if any closed ball in X is compact. We say that a proper geodesic metric space (X, d) is *Gromov hyperbolic* if there is a constant $\delta \geq 0$ such that

$$(x|y)_w \geq (x|z)_w \wedge (z|y)_w - \delta$$

for any $x, y, z, w \in X$, where $a \wedge b := \min\{a, b\}$.

Remark 2.2 The above condition can be rewritten as a 4-point condition:

$$d(x, w) + d(y, z) \leq \max\{d(x, z) + d(y, w), d(x, y) + d(z, w)\} + 2\delta$$

for all $x, w, y, z \in X$.

For a Gromov hyperbolic space X, we recall the definition of Gromov boundary.

Definition 2.3 (1) A sequence $\{x_i\}$ in X is called a *Gromov sequence* if $\left(x_i|x_j\right)_\omega \to \infty$ as $i, j \to \infty$.

(2) Two such sequences $\{x_i\}$ and $\left\{y_j\right\}$ are said to be *equivalent* if $(x_i|y_i)_\omega \to \infty$ as $i \to \infty$.

(3) The *Gromov boundary* $\partial_G X$ of X is defined to be the set of all equivalence classes of Gromov sequences, and $\overline{X}^G = X \cup \partial_G X$ is called the *Gromov closure* of X.

(4) For $a \in X$ and $b \in \partial_G X$, the Gromov product $(a|b)_\omega$ of a and b is defined by

$$(a|b)_\omega = \inf\left\{\liminf_{i\to\infty} (a|b_i)_\omega : \{b_i\} \in b\right\}.$$

(5) For $a, b \in \partial_G X$, the Gromov product $(a|b)_\omega$ of a and b is defined by

$$(a|b)_\omega = \inf\left\{\liminf_{i\to\infty} (a_i|b_i)_\omega : \{a_i\} \in a \text{ and } \{b_i\} \in b\right\}.$$

For a Gromov hyperbolic space X, one can define a class of visual metrics on $\partial_G X$ via the extended Gromov products. For any metric ρ_G in this class, there exist a parameter $\epsilon > 0$ and a base point $\omega \in X$ with

$$\rho_G(a, b) \asymp \exp\left(-\epsilon(a|b)_w\right), \quad \text{for } a, b \in \partial_G X. \tag{2.1}$$

Definition 2.4 Suppose $f : X \to Y$ is a map between metric spaces, and $\lambda \geq 1$, $k \geq 0$ are constants. If for any $x, y \in X$

$$\frac{1}{\lambda} d_X(x, y) - k \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + k,$$

we say that f is an (λ, k)-*quasi-isometry*.

Lemma 2.5 ([3, Part III: Theorem 1.9]) *Suppose (X, d_X) and (Y, d_Y) are geodesic metric spaces and let $f : X \to Y$ be a (λ, k)-quasi-isometric embedding, then (X, d_X) is a Gromov hyperbolic space if (Y, d_Y) is a Gromov hyperbolic space.*

Definition 2.6 Suppose $f : X \to Y$ is a bijection and $\lambda \geq 1$, $\alpha > 0$ are constants. If for distinct $x, y, z \in X$

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta_{\alpha,\lambda}\left(\frac{d_X(x, y)}{d_X(x, z)}\right),$$

we say f is an (α, λ)-power quasisymmetry. Here

$$\eta_{\alpha,\lambda}(w) = \begin{cases} \lambda w^{1/\alpha} & 0 < w < 1 \\ \lambda w^{\alpha} & w \geq 1 \end{cases}$$

Proposition 2.7 ([2, Theorem 6.5]) *Suppose $f : X \to Y$ is a quasi-isometry between Gromov hyperbolic spaces. Then f induces a power quasisymmetry $\bar{f} : \partial_G X \to \partial_G Y$ with respect to the visual metrics. Moreover, $\bar{f}$ is a bi-Hölder map with respect to the visual metrics.*

3 Pseudoconvex Domain of Finite Type in $\mathbb{C}^2$

In this section, we recall some properties of smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ and it refers to [4, 6] for details. We always assume this domain is of finite type of at most m.

3.1 Construction of Polydisks

We introduce a new coordinate near boundary points and construct the polydisks firstly.

Lemma 3.1 *([6, Lemma 2.1]) For any point $\zeta \in U$, where U is a neighborhood of $\partial\Omega$, there is a radius $R > 0$ (independent on ζ) and an injective holomorphic mapping $\Phi_\zeta : B(\zeta, R) \to \mathbb{C}^2$ such that*

(1) Φ_ζ *is of form*

$$\begin{aligned}\Phi_\zeta(z) =&(a_1(\zeta)(z_1-\zeta_1)+a_2(\zeta)(z_2-\zeta_2)\\ &+f_m(\zeta;\langle z-\zeta,L(\zeta)\rangle),\langle z-\zeta,L(\zeta)\rangle)\end{aligned}\tag{3.1}$$

where $a_1(\zeta), a_2(\zeta)$ *depend smoothly on* ζ *and* f_m *is a holomorphic polynomial of degree* m *with coefficients depend smoothly on* ζ *and* $L(z)=(-\frac{\partial r}{\partial z_2}(z),\frac{\partial r}{\partial z_1}(z))$. *Here* r *is the defining function of* Ω.

(2) *The defining function of* Ω *is normalized as* $r=\rho_\zeta\circ\Phi_\zeta$ *with*

$$\rho_\zeta(w)=r(\zeta)+\operatorname{Re} w_1+\sum_{k=2}^{m}P_k(\zeta;w_2)+O(|w_2|^{m+1}+|w||w_1|),$$

where each $P_k(\zeta,\cdot)$ *is a real-valued homogeneous polynomial that has no pure term.*

Now we define the polydisks. For a real-valued k-homogeneous polynomial

$$P(z)=\sum_{j,l\geq 1,\,j+l=k}a_{j,l}z^j\bar{z}^l,$$

we denote

$$\parallel p \parallel:=\max\{|p(e^{i\theta})|:\theta\in\mathbb{R}\}.$$

For any $\zeta\in U$ and a positive number $\delta>0$, we define

$$\tau(\zeta,\delta):=\min\left\{\left(\frac{\delta}{\parallel P_l(\zeta;\cdot)\parallel}\right)^{1/l}:\quad l=2,\ldots,m\right\}$$

and the polydisk

$$R_\delta(\zeta):=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|<\delta,|z_2|<\tau(\zeta,\delta)\}.$$

Denoting

$$Q_\delta(\zeta):=\{z\in B(\zeta,R):\Phi_\zeta(z)\in R_\delta(\zeta)\},$$

we have the following relations.

Lemma 3.2 ([6, Lemma 2.2]) *There exists* $\widehat{C}>0,\hat{\delta}>0$ *such that for all* $0<\delta\leq\hat{\delta}$ *and* $\zeta_1,\zeta_2\in U$, *if* $\zeta_2\in Q_\delta(\zeta_1)$ *we have*

(1) $\zeta_1\in Q_{\widehat{C}\delta}(\zeta_2)$
(2) $Q_\delta(\zeta_1)\subset Q_{\widehat{C}\delta}(\zeta_2)$
(3) $\tau(\zeta_1,\delta)\leq\widehat{C}\tau(\zeta_2,\delta)\leq\widehat{C}^2\tau(\zeta_1,\delta)$
(4) $\tau(\zeta_1,c\delta)\leq c^{1/2}\tau(\zeta_1,\delta)$ *for* $c>1$, $\tau(\zeta_1,c\delta)\leq c^{1/m}\tau(\zeta_1,\delta)$ *for* $0<c\leq 1$.

3.2 The Pseudodistance

Suppose $x, y \in U$, we define that

$$M(x, y) := \{\delta : y \in Q_\delta(x)\},$$

where $Q_\delta(x)$ is defined in Sect. 3.1. Set

$$d'(x, y) = \inf M(x, y)$$

if $M(x, y)$ is not empty (i.e., $|x - y| \leq R$) and $d'(x, y) = +\infty$ otherwise. For $x, y \in U$, denote

$$d(x, y) := \min\{d'(x, y), |x - y|\}.$$

To extend it to Ω, let

$$D(x, y) := \begin{cases} d(x, y) & x, y \in U \\ d_1(x, y) & otherwise \end{cases}$$

where

$$d_1(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$$

Then the following lemma implies that $D(x, y)$ is a pseudodistance on Ω.

Lemma 3.3 ([6, Lemma 3.1]) *There is $C > 0$ such that for any $x, y, z \in \Omega$, we have*

(1) $D(x, y) = 0$ *if and only if* $x = y$
(2) $D(y, x) \leq C D(x, y)$
(3) $D(x, y) \leq C(D(x, z) + D(z, y))$
(4) *If* $x \in U$, $D(x, \pi(x)) \leq C\delta(x)$.
(5) *For* $x, y \in U$ *with* $|x - y| < R$, *then* $d(x, y) \geq C^{-1} d'(x, y)$.

Similar to [10, Lemma 3.4], the inequality (3) above also has the following form.

Lemma 3.4 *There exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ and $x, x_1, \ldots, x_n, y \in \Omega$, we have*

$$D^\epsilon(x, y) \leq 2(D^\epsilon(x, x_1) + D^\epsilon(x_1, x_2) + \cdots + D^\epsilon(x_n, y)).$$

3.3 Local Domains of Comparison

suppose $\zeta \in U$ and

$$\rho_\zeta(w) = r \circ \Phi_\zeta^{-1}(w) = r(\zeta) + \operatorname{Re} w_1 + \sum_{k=2}^{m} P_k(\zeta; w_2) + O(|w_2|^{m+1} + |w||w_1|).$$

For $t > 0$, set

$$J_{\zeta,t}(w) := \left(t^2 + |w_1|^2 + \sum_{k=2}^{m} \| P_k(\zeta; \cdot) \|^2 |w_2|^{2k} \right)^{1/2}$$

and

$$U_{\zeta,t} := \{w : |w| \le R_0, |\rho_\zeta(w)| < s J_{\zeta,t}(w)\}$$

for a suitable $R_0 < R$ and a small constant $s > 0$. Then we have the following lemma.

Lemma 3.5 ([6, Lemma 4.1]) *For sufficiently small $R_0 < R$ and $s > 0$ there exists a smooth real-valued function $H_{\zeta,t}$ in $U_{\zeta,t} \cup \{\rho_\zeta(w) < t\}$ such that*

(1) *For a small constant $\epsilon_0 > 0$, let $\rho_{\zeta,t} = \rho_\zeta + \epsilon_0 H_{\zeta,t}$, then the domain*

$$\Omega_t^\zeta := \{w : \rho_\zeta(w) < t\} \cup \{w \in U_{\zeta,t} : \rho_{\zeta,t}(w) < 0\}$$

is pseudoconvex.

(2) *There exist constants $r_0 > 0, s > 0$ such that for any $\zeta \in \partial\Omega$, $t > 0$ and $|w| < r_0$ the following holds*

$$P(w,t,s) := \left\{(z_1, z_2) : |w_1 - z_1| < s J_{\zeta,t}(w), |w_2 - z_2| < s\tau(\zeta, J_{\zeta,t}(w))\right\} \subset \Omega_t^\zeta$$

provided that $\rho_\zeta(w) < 0$.

3.4 Catlin-Type Metric

Suppose $\Omega = \{z \in \mathbb{C}^2 : r(z) < 0\}$ is a bounded smooth pseudoconvex domain of finite type, and $\xi \in \partial\Omega$ is a point of type m_ξ. We may assume that $\frac{\partial r}{\partial z_2}(\xi) \neq 0$, and define the following vector fields

$$L_1 = \frac{\partial}{\partial z_1} - \left(\frac{\partial r}{\partial z_2}\right)^{-1} \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_2}, \; L_2 = \frac{\partial}{\partial z_2}$$

in a neighborhood of ξ.

Note that $L_1 r = 0$ and L_1, L_2 form a basis of $T_z^{1,0}$ for all z near ξ. For any $j, k > 0$ set

$$\mathcal{L}_{j,k}(z) := \underbrace{L_1 \dots L_1}_{j-1} \underbrace{\bar{L}_1 \dots \bar{L}_1}_{k-1} \partial\bar{\partial} r(L_1, \bar{L}_1)(z)$$

and

$$C_l^{\xi}(z) := \max\{|\mathcal{L}_{j,k}(z)| : j + k = l\}.$$

Let $X = b_1 L_1 + b_2 L_2$ be a holomorphic tangent vector at z. The *Catlin metric* is defined by

$$M_{\xi}(z, X) := \frac{|b_2|}{|r(z)|} + |b_1| \sum_{l=2}^{m_{\xi}} \left(\frac{C_l^{\xi}(z)}{|r(z)|} \right)^{\frac{1}{l}}, \tag{3.2}$$

which implies that

$$K(z, X) \leq C \left(\frac{|(\partial r(z), X)|}{\delta(z)} + \frac{|\langle L(z), X \rangle|}{\tau(z, \delta(z))} \right) \tag{3.3}$$

holds for Kobayashi metric near ξ from [4, Theorem 1].

Moreover, if we set

$$\widetilde{M}_{\xi}(z, X) := \frac{|X_N|}{\delta(z)} + |X_H| \sum_{l=2}^{m_{\xi}} \left(\frac{C_l^{\xi}(z)}{\delta(z)} \right)^{\frac{1}{l}}, \tag{3.4}$$

then $\widetilde{M}_{\xi}(z, X) \asymp K(z, X)$ in a neighborhood of ξ by [7, Lemma 2.2].

By choosing an open neighborhood U_i of ξ_i, which form a finite cover of $\partial\Omega$, then there exists a constant $\epsilon > 0$ such that

$$N_{\epsilon}(\partial\Omega) \subset \bigcup_{i=1}^{s} U_i.$$

For $z \in \Omega \cap N_{\epsilon}(\partial\Omega)$, we denote $I_z = \{i : z \in U_i\}$ and set

$$\widetilde{M}(z, X) := \max_{z \in I_z} \{\widetilde{M}_{\xi}(z, X)\}.$$

Since it is upper semi-continuous, we can define a global Finsler metric in Ω by

$$\widetilde{K}(z, X) := K(z, X) S(z, X)$$

with a positive function $S(z, X) \asymp 1$ and $\widetilde{K}(z, X) = \widetilde{M}(z, X)$ for $z \in \Omega \cap N_{\epsilon_0}(\partial\Omega)$. And it implies that $\widetilde{K}(z, X) \asymp K(z, X)$ in Ω. We may call it *Catlin-type metric* and denote $d_{\widetilde{K}}$ the distance associated to this metric.

Note that the Catlin-type distance statisfies the following estimate.

Lemma 3.6 ([7, Lemma 2.3]) *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then there exists a constant $\epsilon > 0$ such that, for any $x, y \in N_\epsilon(\partial\Omega)$, we have*

$$d_{\widetilde{K}}(x, y) \geq \left|\log\left(\frac{\delta(y)}{\delta(x)}\right)\right|. \tag{3.5}$$

Moreover, if $\pi(x) = \pi(y)$, there is

$$d_{\widetilde{K}}(x, y) = \left|\log\left(\frac{\delta(y)}{\delta(x)}\right)\right|. \tag{3.6}$$

4 Estimate of Catlin-Type Distance

In this section, we prove Theorems 1.2 and 1.3. We firstly prove Theorem 1.2 and state it again for convenience.

Theorem 4.1 *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type, then there is a constant $C > 0$ such that*

$$g(x, y) - C \leq d_{\widetilde{K}}(x, y) \leq g(x, y) + C \tag{4.1}$$

for any $x, y \in \Omega$.

Similar to [10, Lemma 4.4], we will prove the following lemma firstly.

Lemma 4.2 *For any $c > 0$, there is a constant $c_0 > 0$ such that for $x, y \in U \cap \Omega$, if $y \in \partial Q_{c\delta(x)}(x)$ then we have*

$$d_{\widetilde{K}}(x, y) \geq c_0. \tag{4.2}$$

Proof Since Catlin-type metric is comparable to Kobayashi metric, we just need to prove the lemma for Kobayashi distance. If $c \geq 4\widehat{C}$, where $\widehat{C}$ is the constant in Lemma 3.2, then by Lemma 6.1 in [6] we have

$$y \notin Q_{4\widehat{C}\delta(x)}(x) \supset Q_{2\widehat{C}\delta(x)}(\pi(x)).$$

And Lemma 5.3 in [6] shows that if $b \notin Q_{2\delta(a)}(\pi(a))$, then

$$d_K(a, b) \geq c_0.$$

It deduces that the lemma holds for $c \geq 4\widehat{C}$.

If $c < 4\widehat{C}$, then we have that

$$\Phi_x(y) \in \{(z_1, z_2) : |z_1| < 4\widehat{C}\delta(x), |z_2| < \tau(x, 4\widehat{C}\delta(x))\} \subset P(w_x, t, s)$$

with a number $s > 0$ which does not depend on x, y and $t = 4\widehat{C}\delta(x)$, $w_x = \Phi_{\pi(x)}(x)$.

We set

$$f(z_1, z_2) = \frac{1}{C\delta(x)}(\Phi_x \circ \Phi^{-1}_{\pi(x)}(z_1, z_2))_1$$

such that $\| f \|_\infty \leq 1$ with suitable C, which is independent of x, y. To see this we observe that

$$\Phi_x \circ \Phi^{-1}_{\pi(x)}(z_1, z_2) = (\Phi_x - \Phi_{\pi(x)}) \circ \Phi^{-1}_{\pi(x)}(z_1, z_2) + (z_1, z_2),$$

and hence for some constant $C_1 > 0$

$$|f(z_1, z_2)| \leq \frac{|(\Phi_x - \Phi_{\pi(x)}) \circ \Phi^{-1}_{\pi(x)}(z_1, z_2)| + |z_1|}{C\delta(x)} \leq C_1 + \frac{|z_1|}{C\delta(x)}.$$

Indeed, there exists a constant $C_2 > 0$ such that

$$(\Phi_x - \Phi_{\pi(x)}) \circ \Phi^{-1}_{\pi(x)}(z_1, z_2) \leq C_2|x - \pi(x)| = C_2\delta(x)$$

from equality (3.1) in Lemma 3.1. And on $P(w_x, t, s)$ the following holds

$$|z_1| \leq |z_1 - (w_x)_1| + |(w_x)_1| \leq sJ_{\pi(x),t}(w_x) + t \leq C_3\delta(x)$$

with $f(w_x) = 0$ and constant $C_3 > 0$. Moreover, Lemma 5.2 in [6] implies that there exists a function $\tilde{f} \in H^\infty(\Omega)$ such that $||\tilde{f}||_\infty \leq L^*||f||_\infty$ and $\tilde{f}(w_x) = f(w_x)$, $\tilde{f}(w_y) = f(w_y)$. It means that

$$d_K(x, y) \geq d_C(x, y) \geq d_\Delta(0, \frac{1}{L^*}\tilde{f}(w_y)) \geq \frac{1}{2}\log\left(1 + \frac{|\Phi_x(y)_1|^2}{(L^*C\delta(x))^2}\right). \tag{4.3}$$

Since $y \in \partial Q_{c\delta(x)}(x)$, then $|\Phi_x(y)_1| = c\delta(x)$ or $|\Phi_x(y)_2| = \tau(x, c\delta(x))$ holds. If $|\Phi_x(y)_1| = c\delta(x)$ holds, from (4.3) we obtain $d_{\widetilde{K}}(x, y) \geq c_0$. If $|\Phi_x(y)_2| = \tau(x, c\delta(x))$ holds, by choosing

$$f(z_1, z_2) = \frac{1}{C\tau(x, \delta(x))}(\Phi_x \circ \Phi^{-1}_{\pi(x)}(z_1, z_2))_2$$

with suitable C we get that

$$d_K(x, y) \geq d_C(x, y) \geq d_\Delta(0, \frac{1}{L^*}f(w_y)) \geq \frac{1}{2}\log\left(1 + \frac{|\Phi_x(y)_2|^2}{(L^*C\tau(x, \delta(x)))^2}\right). \tag{4.4}$$

It implies $d_{\widetilde{K}}(x, y) \geq c_0$ whenever $y \in \partial Q_{c\delta(x)}(x)$. □

Next we prove the following result which gives the upper bound $d_{\widetilde{K}}(x, y)$ when $\delta(x) = \delta(y)$ and $D(x, y) \leq c\delta(x)$ with a constant $c > 0$.

Lemma 4.3 *After shrinking U, then for any $c > 0$ there exist a constant $C > 0$ such that, for any $x, y \in \Omega \cap U$ with $\delta(x) = \delta(y)$ and $D(x, y) \leq c\delta(x)$, the following holds*

$$d_{\widetilde{K}}(x, y) \leq C. \tag{4.5}$$

Proof We just need to prove the lemma for Kobayashi distance d_K. If $D(x, y) = |x - y|$, then Theorem 3.5 in [7] implies that there exists a constant $C_1 > 0$ such that

$$d_{\widetilde{K}}(x, y) \leq 2 \log \frac{|x - y| + \delta(x) \vee \delta(y)}{\sqrt{\delta(x)\delta(y)}} + C_1.$$

And it implies the lemma.

If $D(x, y) = d'(x, y)$. From [6, Main Theorem 2.1], we have

$$d_K(x, y) \leq C_2(\rho(x, y) + \rho(y, x))$$

for some constant $C_2 > 0$ with

$$\rho(x, y) = \log(1 + \frac{d'(x, y)}{\delta(x)} + \frac{|\Phi_x(y)_2|}{\tau(x, \delta(x))}).$$

Since $d'(x, y) \leq c\delta(x)$ and according to (4) in Lemma 3.2, there exists a constant $C_3 > 0$ such that $|\Phi_x(y)_2| \leq \tau(x, D(x, y)) \leq \tau(x, c\delta(x)) \leq C_3\tau(x, \delta(x))$. Consequently, we can conclude that $\rho(x, y) \leq C_4$, $\rho(y, x) \leq C_4$ with some $C_4 > 0$. It completes the proof. □

Proof of Theorem 4.1. Now we proceed to prove Theorem 4.1, which is similar to [1, Theorem 1.1]. Let $N := N_{\epsilon_0}(\partial\Omega) \cap \Omega$ be a neighborhood such that Lemmas 2.1, 4.2 and 4.3 hold. Denote K the closure of $\Omega \setminus N$, then K is compact in Ω. In the following dicussion, we will denote by C positive constants only depending on ϵ_0 and the various other constants that are associated with Ω. It is important to note that the specific value of C is not relevant and may change even within the same line.

(1) If $x, y \in K$, then

$$0 \leq g(x, y) \leq C, \ 0 \leq d_{\widetilde{K}}(x, y) \leq C.$$

Hence inequality (4.1) is true.

(2) If $x \in K, \ y \in N$ or $y \in K, \ x \in N$. We may assume that $x \in K, \ y \in N$. Then

$$\log \frac{1}{\delta(y)} - C \leq g(x, y) \leq \log \frac{1}{\delta(y)} + C.$$

To get the upper bound of $d_{\widetilde{K}}$, we set $y' = \pi(y) - \epsilon_0\vec{n}(\pi(y))$. We know that $y' \in K, \ \pi(y') = \pi(y)$. From Lemma 3.6, we know that

$$d_{\widetilde{K}}(y, y') = \log \frac{1}{\delta(y)} + C.$$

Since $x, y' \in K$, then $d_{\widetilde{K}}(x, y') \leq C$. It implies that $d_{\widetilde{K}}(x, y) \leq d_{\widetilde{K}}(x, y') + d_{\widetilde{K}}(y', y) \leq \log \frac{1}{\delta(y)} + C$. Hence we get the right half of inequality (4.1).

To obtain lower bound for $d_{\widetilde{K}}(x, y)$. Let γ be an arbitrary piecewise C^1-smooth curve in Ω joining y and x. Then there is a first point x' on the curve with $x' \in K$, which means that $\delta(x') = \epsilon_0$. Suppose α is the subcurve of γ with endpoints y and x' and let $L_{\widetilde{K}}(\gamma)$ denote the length of the curve γ with respect to the Finlser metric $\widetilde{K}$, then we have by Lemma 3.6

$$L_{\widetilde{K}}(\gamma) \geq L_{\widetilde{K}}(\alpha) \geq \log \frac{1}{\delta(y)} - C.$$

Taking the infimum for γ, we obtain

$$d_{\widetilde{K}}(x, y) \geq \log \frac{1}{\delta(y)} - C.$$

It completes the proof.

(3) Now we may assume that $x, y \in N$. First we estimate the upper bound of $d_{\widetilde{K}}$.

Case 1. $D(x, y) \leq \delta(x) \vee \delta(y)$.

We may assume that $\delta(y) \geq \delta(x)$. Then

$$\log \frac{\delta(y)}{\delta(x)} \leq g(x, y) \leq \log \frac{\delta(y)}{\delta(x)} + C.$$

Let $x' = \pi(x) - \delta(y) \cdot \vec{n}(\pi(x))$, by Lemma 3.6 we get

$$d_{\widetilde{K}}(x, x') = \log \frac{\delta(y)}{\delta(x)}.$$

From Lemma 4.3 we obtain $d_{\widetilde{K}}(x', y) \leq C$, which means that

$$d_{\widetilde{K}}(x, y) \leq d_{\widetilde{K}}(x, x') + d_{\widetilde{K}}(x', y) \leq g(x, y) + C.$$

Case 2. $D(x, y) > \delta(x) \vee \delta(y)$.

We have that

$$2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} \leq g(x, y) \leq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} + C.$$

Let $x' = \pi(x) - h_0 \vec{n}(\pi(x))$ and $y' = \pi(y) - h_0 \vec{n}(\pi(y))$ with $h_0 = D(x, y) \wedge \epsilon_0$. Note that $\delta(x') = \delta(y') = h_0 \leq D(x, y)$. Then the following holds

$$d_{\widetilde{K}}(x', x) + d_{\widetilde{K}}(y', y) \leq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}}.$$

Therefore, we only need to estimate the upper bound of $d_{\widetilde{K}}(x', y')$. If $h_0 = \epsilon_0$, then both x' and y' lie in K, hence $d_{\widetilde{K}}(x', y') \leq C$. We may assume that $h_0 = D(x, y)$.

From Lemma 3.3 we get that $D(x', x) \leq CD(x, y)$ and $D(y', y) \leq CD(x, y)$, which means $D(x', y') \leq CD(x, y)$. By Lemma 4.3, we have

$$d_{\widetilde{K}}(x', y') \leq C.$$

It deduces that

$$d_{\widetilde{K}}(x, y) \leq g(x, y) + C.$$

Now we estimate the lower bound of $d_{\widetilde{K}}$. We may assume that $\delta(y) \geq \delta(x)$. □

Case 1. $D(x, y) \leq \delta(x) \vee \delta(y)$.

We have that

$$\log \frac{\delta(y)}{\delta(x)} \leq g(x, y) \leq \log \frac{\delta(y)}{\delta(x)} + C$$

and by Lemma 3.6

$$d_{\widetilde{K}}(x, y) \geq \log \frac{\delta(y)}{\delta(x)} - C,$$

which is the desired inequality.

Case 2. $D(x, y) > \delta(x) \vee \delta(y)$.

We know that

$$2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} \leq g(x, y) \leq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} + C.$$

Let $\gamma : [0, 1] \to \Omega$ be an arbitrary curve joining x and y. Define $H = \max_{z \in \gamma} \delta(z)$. There exists a constant $t_0 \in [0, 1]$ such that $\delta(\gamma(t_0)) = H$. We consider two curves $\gamma_1 = \gamma|_{[0,t_0]}$ and $\gamma_2 = \gamma|_{[t_0,1]}$. There are two possibilities.

(a) If $H \geq h_0$ with $h_0 = D(x, y) \wedge \epsilon_0$, then

$$L_{\widetilde{K}}(\gamma_1) \geq \log \frac{h_0}{\delta(x)}, \; L_{\widetilde{K}}(\gamma_2) \geq \log \frac{h_0}{\delta(y)}.$$

Thus we get that

$$L_{\widetilde{K}}(\gamma) \geq \log \frac{h_0^2}{\delta(x)\delta(y)} \geq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} - C$$

with $h_0 \geq CD(x, y)$.

(b) If $H < h_0$.
Since $\delta(x) \le H$, there exists a integer $k \in \mathbb{N}$ such that

$$\frac{H}{2^k} < \delta(x) \le \frac{H}{2^{k-1}}.$$

We consider the curve γ_1 and define $0 = s_0 \le s_1 < \cdots < s_k \le t_0$ as follows. Let

$$s_j = \max\left\{s \in [0, t_0] : \delta(\gamma(s)) = \frac{H}{2^{k-j}}\right\}$$

for $j = 1, 2 \ldots, k$. Setting $z_j = \gamma(s_j)$, we note that

$$1 \le \frac{\delta(z_j)}{\delta(z_{j-1})} \le 2$$

for $j = 1, \ldots, k$.

Let $\epsilon = \epsilon_0$ in Lemma 3.4 and denote $c_1 = \left(\frac{1 - 1/2^{\epsilon}}{16}\right)^{\frac{1}{\epsilon}}$. There are two cases:

(i) There is an index $l \in \{1, \ldots, k\}$ such that

$$D(z_{l-1}, z_l) > c_1 \frac{D(x, y)}{2^{k-l}}.$$

Note that for $t \in [s_{l-1}, s_l]$ one have

$$\delta(\gamma(t)) \le \frac{H}{2^{k-l}}.$$

Denote $p = z_{l-1}$ and $q = z_l$. Since

$$D(p, q) > c_1 \frac{D(x, y)}{2^{k-l}} \ge c_1 \delta(p), \tag{4.6}$$

we have that

$$q \notin Q_{c_1\delta(p)}(p).$$

It means $\gamma \cap \partial Q_{c_1\delta(p)}(p) \ne \emptyset$. Hence the following set

$$S^{p,q} = \{m \in \mathbb{N} : \exists u_1, \ldots, u_m \text{ sothat } s_{l-1} = u_0 < u_1 < \cdots < u_m \le s_l, \\ \gamma(u_j) \in \partial Q_{c_1\delta(p_{j-1})}(p_{j-1}), 1 \le j \le m\}$$

is not empty with $p_{j-1} = \gamma(u_{j-1})$.

By Lemma 4.2 there exists $c_0 > 0$ such that

$$L_{\widetilde{K}}(\gamma|_{[s_{l-1},s_l]}) \geq \sum_{j=1}^{m} d_{\widetilde{K}}(p_{j-1}, p_j) \geq c_0 m,$$

we know $S^{p,q}$ is a finite set. Denote $m_0 = \max S^{p,q}$, then

$$\begin{aligned} D^{\epsilon}(p,q) &\leq 2\left(\sum_{j=1}^{m_0} D^{\epsilon}(p_{j-1}, p_j) + D^{\epsilon}(p_{m_0}, q)\right) \\ &\leq 2c_1^{\epsilon}(m_0+1)\left(\frac{H}{2^{k-l}}\right)^{\epsilon}. \end{aligned}$$

Hence one obtain that

$$m_0 \geq \frac{1}{2}\left(\frac{D(x,y)}{H}\right)^{\epsilon} - 1,$$

which means

$$L_{\widetilde{K}}(\gamma|_{[s_{l-1},s_l]}) \geq \frac{c_0}{2}\left(\frac{D(x,y)}{H}\right)^{\epsilon} - c_0.$$

Let $t_1 = s_k \leq t_0$, then

$$\begin{aligned} L_{\widetilde{K}}(\gamma|_{[0,t_1]}) &= L_{\widetilde{K}}(\gamma|_{[0,s_{l-1}]}) + L_{\widetilde{K}}(\gamma|_{[s_{l-1},s_l]}) + L_{\widetilde{K}}(\gamma|_{[s_l,s_k]}) \\ &\geq \log\left(\frac{\delta(z_{l-1})}{\delta(z_0)}\right) + \frac{c_0}{2}\left(\frac{D(x,y)}{H}\right)^{\epsilon} + \log\left(\frac{\delta(z_k)}{\delta(z_l)}\right) - C \\ &\geq \log\left(\frac{H}{\delta(x)}\right) + \frac{c_0}{2}\left(\frac{D(x,y)}{H}\right)^{\epsilon} - C. \end{aligned}$$

(ii) If the following holds

$$D(z_{j-1}, z_j) \leq c_1 \frac{D(x,y)}{2^{k-j}}$$

for all $j = 1, \ldots, k$, then

$$D^{\epsilon}(x, \gamma(t_1)) \leq 2\sum_{j=1}^{k} D^{\epsilon}(z_{j-1}, z_j) \leq \frac{2c_1^{\epsilon}}{1 - 1/2^{\epsilon}} D^{\epsilon}(x,y) \leq \frac{1}{8} D^{\epsilon}(x,y).$$

Moreover, we also have

$$L_{\widetilde{K}}(\gamma|_{[0,t_1]}) \geq \log\frac{H}{\delta(x)}.$$

Thus there are two cases:

$$L_{\widetilde{K}}(\gamma|_{[0,t_1]}) \geq \log\left(\frac{H}{\delta(x)}\right) + C\left(\frac{D(x,y)}{H}\right)^{\epsilon} - C \tag{4.7}$$

or

$$L_{\widetilde{K}}(\gamma|_{[0,t_1]}) \geq \log\frac{H}{\delta(x)} \text{ and } D^{\epsilon}(x,\gamma(t_1)) \leq \frac{1}{8}D^{\epsilon}(x,y). \tag{4.8}$$

Applying same consideration to γ_2 we can find $t_2 \in [t_0, 1]$ such that one of the following alternatives holds

$$L_{\widetilde{K}}(\gamma|_{[t_2,1]}) \geq \log\left(\frac{H}{\delta(y)}\right) + C\left(\frac{D(x,y)}{H}\right)^{\epsilon} - C \tag{4.9}$$

or

$$L_{\widetilde{K}}(\gamma|_{[t_2,1]}) \geq \log\frac{H}{\delta(y)} \text{ and } D^{\epsilon}(y,\gamma(t_2)) \leq \frac{1}{8}D^{\epsilon}(x,y). \tag{4.10}$$

Suppose (4.8) and (4.10) hold, we have

$$\begin{aligned} D^{\epsilon}(\gamma(t_1),\gamma(t_2)) &\geq \frac{1}{2}D^{\epsilon}(x,y) - D^{\epsilon}(x,\gamma(t_1)) - D^{\epsilon}(\gamma(t_2),y) \\ &\geq \frac{1}{4}D^{\epsilon}(x,y) \geq \frac{1}{4}H^{\epsilon} \end{aligned}$$

which means that

$$D(\gamma(t_1),\gamma(t_2)) \geq \frac{D(x,y)}{4^{\frac{1}{\epsilon}}} \geq \frac{H}{4^{\frac{1}{\epsilon}}} = \frac{\delta(t_1)}{4^{\frac{1}{\epsilon}}}.$$

By using the same discussion for (4.6) in (i), we deduce that

$$L_{\widetilde{K}}(\gamma|_{[t_1,t_2]}) \geq C\left(\frac{D(x,y)}{H}\right)^{\epsilon} - C,$$

which implies that

$$\begin{aligned} L_{\widetilde{K}}(\gamma) &= L_{\widetilde{K}}(\gamma|_{[0,t_1]}) + L_{\widetilde{K}}(\gamma|_{[t_1,t_2]}) + L_{\widetilde{K}}(\gamma|_{[t_2,1]}) \\ &\geq 2\log\frac{H}{\sqrt{\delta(x)\delta(y)}} + C\left(\frac{D(x,y)}{H}\right)^{\epsilon} - C \end{aligned}$$

and the last inequality holds for other cases. Set

$$h(t) = 2\log\frac{t}{\sqrt{\delta(x)\delta(y)}} + C\left(\frac{D(x,y)}{t}\right)^{\epsilon}$$

we know that h has a minimum if

$$t = \left(\frac{C\epsilon}{2}\right)^{1/\epsilon} D(x, y).$$

And it implies that

$$L_{\widetilde{K}}(\gamma) \geq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} - C.$$

If we take infimum for γ, then we get that

$$d_{\widetilde{K}}(x, y) \geq 2 \log \frac{D(x, y)}{\sqrt{\delta(x)\delta(y)}} - C$$

and the proof is completed. □

Proof of Theorem 1.3. The following proof is similar to [1, Theorem 1.4] and we present it here for the sake of completeness.

Suppose $r_{ij} \geq 0$ are given numbers such that $r_{ij} \leq C r_{ji}$ and $r_{ij} \leq C(r_{ik} + r_{kj})$ for $i, j, k \in \{1, 2, 3, 4\}$ with a constant $C \geq 1$. Then we have $r_{12}r_{34} \leq 4C^4((r_{13}r_{24}) \vee (r_{14}r_{23}))$. Indeed, we assume that r_{13} is the smallest of the quantities r_{ij} appearing on the right hand side of this inequality. Then we know that

$$r_{12} \leq C(r_{13} + r_{32}) \leq C(C + 1)r_{23} \leq 2C^2 r_{23}$$

and $r_{34} \leq C(r_{31} + r_{14}) \leq 2C^2 r_{14}$. It deduces the inequality.

Set $x_i \in \Omega$ $(i = 1, 2, 3, 4)$ and $r_{ij} = d_{ij} + h_i \vee h_j$ where $d_{ij} = D(x_i, x_j)$, $h_i = \delta(x_i)$, then we have $r_{ij} \leq C_1 r_{ji}$ and $r_{ij} \leq C_1(r_{ik} + r_{kj})$ for some constant $C_1 \geq 1$. It implies that

$$r_{12}r_{34} \leq 4C_1^2((r_{13}r_{24}) \vee (r_{14}r_{23}))$$

and

$$\begin{aligned}&(d_{12} + h_1 \vee h_2)(d_{34} + h_3 \vee h_4)\\ &\leq 4C_1^2(((d_{13} + h_1 \vee h_3)(d_{24} + h_2 \vee h_4)) \vee ((d_{14} + h_1 \vee h_4)(d_{23} + h_2 \vee h_3))).\end{aligned}$$

From inequality (4.1), we obtain

$$\begin{aligned}d_{\widetilde{K}}(x_1, x_2) + d_{\widetilde{K}}(x_3, x_4) \leq (d_{\widetilde{K}}(x_1, x_3) + d_{\widetilde{K}}(x_2, x_4)) \vee (d_{\widetilde{K}}(x_1, x_4)\\ + d_{\widetilde{K}}(x_2, x_3)) + C',\end{aligned}$$

for some constant $C' > 0$ independent of points. By Remark 2.2, this inequality is equivalent to the Gromov hyperbolicity of the space $(\Omega, d_{\widetilde{K}})$. □

Acknowledgements The authors would like to thank Professor Jinsong Liu for many precious suggestions. We would also like to thank the referee for a careful reading and valuable comments.

Data Availability No data was used for the research described in the article.

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Annali di Matematica Pura ed Applicata (1923 -) (2024) 203:2785–2799
https://doi.org/10.1007/s10231-024-01466-8

The Gehring–Hayman type theorem on pseudoconvex domains of finite type in $\mathbb{C}^2$

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Received: 16 October 2023 / Accepted: 13 May 2024 / Published online: 5 June 2024
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Abstract
In this paper, we obtain the Gehring–Hayman type theorem on smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. As an application, we provide a quantitative comparison between global and local Kobayashi distances near a boundary point for these domains.

Keywords Quasi-geodesic · Kobayashi distance · pseudoconvex domain · Gromov hyperbolicity

Mathematics Subject Classification Primary 32T25

1 Introduction

In the complex plane $\mathbb{C}$, the classical Gehring-Hayman Theorem on planar domains was originally established by Gehring-Hayman in their work [9]. The theorem was stated as follows.

The first author was supported by NSFC (No. 12326407); the third author was supported by the Young Scientist Program of the Ministry of Science and Technology of China (No. 2021YFA1002200) and NSFC (No. 12201059).

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Theorem 1.1 *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected planar domain, then there exists a constant $C > 0$ such that for any $x, y \in \Omega$,*

$$L([x, y]) \leq CL(\gamma),$$

where $[x, y]$ is the hyperbolic geodesic connecting x and y, and $\gamma \subset \Omega$ is any curve with end points x and y, and $L(\gamma)$ denotes the Euclidean length of γ.

Later, in the real space $\mathbb{R}^n$, $n \geq 2$, Gehring-Osgood [10] generalized the result to quasihyperbolic geodesics on uniform domains by considering the quasihyperbolic metrics. Moreover, it was used to characterize the Gromov hyperbolicity of domains in $\mathbb{R}^n$(see [2, 4]).

On the other hand, in the complex space $\mathbb{C}^n$, $n \geq 2$, Liu-Wang-Zhou [16] extended this result to m-convex domains by considering the Kobayashi metrics. Subsequently, Nikolov-Ökten [18] obtained similar results for more general domains and improved the index for m-convex domains. Recently, in our work [15], we proved that the index can be $\frac{1}{m}$. Furthermore, Kosiński-Nikolov-Thomas [13] showed that for bounded strongly pseudoconvex domains with $C^{2,\alpha}$-smooth boundary, the index is 1. In this paper, we get the Gehring-Hayman type theorem for smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. The main theorem is as follows.

Theorem 1.2 *Let $\Omega \subset \mathbb{C}^2$ be is a smoothly bounded pseudoconvex domain of finite type m. Then there exists a constant $C = C(\lambda) > 0$ such that for every $x, y \in \Omega$,*

$$L(\gamma) \leq C|x - y|^{\frac{1}{m}}, \tag{1.1}$$

where γ is a Kobayashi λ-quasi-geodesic connecting x and y with $\lambda \geq 1$. Moreover, denoting $H_\gamma = \max_{z\in\gamma} \delta(z)$, there exists a constant $C' = C'(\lambda) > 0$ such that

$$H_\gamma^{\frac{1}{m}} \geq C' L(\gamma). \tag{1.2}$$

Remark 1.3 (1) Observing that for m-convex domains that are Gromov hyperbolic, the proof of λ-quasi-geodesic in [15, Theorem 3.3] can be substituted with the proof of Theorem 1.2.

(2) It is worth noting that, inequality (1.2) is also valid for smoothly bounded convex domains of finite type in $\mathbb{C}^n$, and Wang has obtained a similar estimate in [22, Lemma 5.6]. Moreover, for bounded strongly pseudoconvex domains with C^2-smooth boundary, the exponents are reduced to $1/2$ when utilizing Gromov hyperbolicity. This observation extends Corollary 12 in [18] and further demonstrates the validity of [12, Proposition 3] under lower boundary regularity assumptions.

In order to prove the theorem, we first illustrate the estimate in [15, Theorem 1.3] is valid for λ-quasi-geodesic when the domain is Gromov hyperbolic.

Theorem 1.4 *Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a bounded Gromov hyperbolic domain respect to a complete Finsler distance d_F. Suppose that there exist constants $C_1, C_2, \alpha > 0, 1 > \beta > 0$ such that, for all $x, y \in \Omega$,*

$$d_F(x, y) \geq \alpha \left| \log \left(\frac{\delta(x)}{\delta(y)} \right) \right| - C_1,$$

and the Finsler metric

$$F(z, X) \geq \frac{C_2 |X|}{\delta(z)^\beta}, \quad \forall z \in \Omega, \ 0 \neq X \in \mathbb{C}^n.$$

For each $x, y \in \Omega$, let $L(\gamma)$ denote the Euclidean length of a Finsler λ-quasi-geodesic γ connecting x and y in Ω. Then there exists a constant $C = C(\lambda) > 0$ such that

$$d_F(x, y) \geq 2\alpha \log \left(\frac{L(\gamma)^{\frac{1}{\beta}}}{\sqrt{\delta(x)\delta(y)}} \right) - C.$$

As an application, we provide a quantitative comparison between global and local Kobayashi distances near a boundary point on smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. It partially extends the result in [21, Corollary 1.5].

Theorem 1.5 *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain of finite type m. Suppose that $\Omega \cap U$ is connected in a neighborhood U of $\xi \in \partial\Omega$. Then there exists a neighborhood V of ξ with $V \subset\subset U$ and constants $C > 0, A \geq 1$ such that, for any $x, y \in \Omega \cap V$,*

$$d_{K_\Omega}(x, y) \leq d_{K_{\Omega\cap U}}(x, y) \leq d_{K_\Omega}(x, y) + C|x - y|^{\frac{2}{Am}}, \tag{1.3}$$

and

$$1 \leq \frac{d_{K_{\Omega\cap U}}(x, y)}{d_{K_\Omega}(x, y)} \leq 1 + C \left(\delta_\Omega(x) \wedge \delta_\Omega(y) + |x - y|^{\frac{1}{m}} \right)^{\frac{2}{A}} \text{ for } x \neq y \in \Omega \cap V. \tag{1.4}$$

Remark 1.6 The quantitative localization of Kobayashi distance on complex domains was first derived by Nikolov-Thomas [19, Theorem 1.1] near a locally $\mathbb{C}$-convexifiable point of finite type on the boundary. And inspired by their results, we relaxed the boundary regularity to Dini-smooth in [15, Theorem 1.9]. Here we actually expand the boundary convexity for the result in $\mathbb{C}^2$.

The paper is organized as follows. In Sect. 2 we will recall some definitions and preliminary results. In Sect. 3 we give the proof of Theorems 1.4 and 1.2. In Sect. 4 we will illustrate the proof of Theorem 1.5.

2 Preliminaries

2.1 Notation

(1) For $z \in \mathbb{C}^n$, let $|\cdot|$ denote the standard Euclidean norm, and let $|z_1 - z_2|$ denote the standard Euclidean distance of $z_1, z_2 \in \mathbb{C}^n$.
(2) Given an open set $\Omega \subsetneq \mathbb{C}^n$, $x \in \Omega$ and $v \in \mathbb{C}^n$, denote

$$\delta(x) \text{ or } \delta_\Omega(x) = \inf\{|x - \xi| : \xi \in \partial\Omega\}.$$

(3) Recall that, for real numbers a, b, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.
(4) For any curve γ, denote its Euclidean length and Finsler length by $L(\gamma)$ and $L_F(\gamma)$ respectively.
(5) For functions f, g, write $f \lesssim g$ if there exists $C > 0$ such that $f \leq Cg$, and we write

$$f \asymp g \text{ if } f \lesssim g \text{ and } g \lesssim f.$$

(6) Recall that, for any $p \in \partial\Omega$, the complex tangent space is given by

$$H_p\partial\Omega = \left\{ X \in \mathbb{C}^n : \langle \bar{\partial} r(p), X \rangle = 0 \right\},$$

where

$$\bar{\partial} r(p) = \left(\frac{\partial r}{\partial \bar{z}_1}(p), \ldots, \frac{\partial r}{\partial \bar{z}_n}(p) \right).$$

Here r is a local defining function of Ω whose gradient does not vanish at p, and the standard Hermitian product in $\mathbb{C}^n$ is $\langle X, Y \rangle = \sum_{k=1}^n X_k \overline{Y}_k$. Therefore, for any vector $0 \neq X \in \mathbb{C}^n$ it has a unique orthogonal decomposition $X = X_H + X_N$ with $X_H \in H_p \partial\Omega$ and $X_N \in N_p \partial\Omega$. Here $N_p \partial\Omega$ is the complex one-dimensional subspace of $\mathbb{C}^n$ orthogonal to $H_p \partial\Omega$.

(7) In general, we will denote by C positive constants in the proofs. It is important to note that the specific value of C is not relevant and may change from line to line.

2.2 The Finsler metric

Given a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, a Finsler metric on Ω is an upper semi-continuous map $F : \Omega \times \mathbb{C}^n \to [0, +\infty)$ with $F(z; tX) = |t| F(z; X)$ for any $z \in \Omega$, $t \in \mathbb{C}$ and $X \in \mathbb{C}^n$. The distance function d_F associated with F is defined by

$$d_F(x, y) = \inf\{F\text{-length}(\gamma) : \gamma : [0, 1] \to \Omega \text{ is a piecewise } C^1\text{-smooth curve with } \gamma(0) = x, \gamma(1) = y\},$$

where

$$F\text{-length}(\gamma) = \int_0^1 F(\gamma(t); \dot{\gamma}(t)) dt.$$

A very important Finsler metric in several complex variables is the Kobayashi metric. For a domain $\Omega \subset \mathbb{C}^n$, the (infinitesimal) Kobayashi metric is defined by

$$K_\Omega(x; v) = \inf \{|\xi| : f \in \mathrm{Hol}(\mathbb{D}, \Omega), \text{ with } f(0) = x, d(f)_0(\xi) = v\}.$$

For convenience, we denote by d_{K_Ω} the Kobayashi distance associated with the Kobayashi metric K_Ω, and sometimes we may omit the subscript Ω. The main property of the Kobayashi distance is that it is contracted by holomorphic maps. That is, if $f : \Omega_1 \to \Omega_2$ is a holomorphic map, then

$$\forall z, w \in \Omega_1 \quad d_{K_{\Omega_2}}(f(z), f(w)) \leqslant d_{K_{\Omega_1}}(z, w).$$

Recall that a C^1-smooth boundary point p of a domain Ω in $\mathbb{C}^n$ is said to be Dini-smooth, if the outer unit normal vector $\vec{n}$ to $\partial\Omega$ near p is a Dini-continuous function. This means that there exists a neighborhood U of p with

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty,$$

where

$$\omega(t) = \omega(\vec{n}, \partial\Omega \cap U, t) := \sup\{|\vec{n}(x) - \vec{n}(y)| : |x - y| < t, \ \ x, y \in \partial\Omega \cap U\}$$

is the respective modulus of continuity. Note that Dini-smooth is a weaker condition than $C^{1,\epsilon}$-smooth. Here a Dini-smooth domain means that each boundary point of Ω is a Dini-smooth point.

Then we have the following upper bound of Kobayashi distance.

Lemma 2.1 [17, Corollary 8] *Let Ω be a Dini-smooth bounded domain in $\mathbb{C}^n$ and $x, y \in \Omega$. Then there exists a constant $C > 1 + \sqrt{2}/2$ such that*

$$d_K(x, y) \leq \log\left(1 + \frac{C|x - y|}{\sqrt{\delta_\Omega(x)\delta_\Omega(y)}}\right).$$

2.3 Catlin-type metric

Suppose that $\Omega = \{z \in \mathbb{C}^2 : r(z) < 0\}$ is a smoothly bounded pseudoconvex domain of finite D'Angelo type, and $\xi \in \partial\Omega$ is a point of type m_ξ. By using a rotation of the canonical coordinates, we can arrange that the normal direction to $\partial\Omega$ at ξ is given by the $\operatorname{Im} z_1$-axis. Supposing that $\xi = 0$, and by using Implicit function theorem, we obtain a local defining function of the form $r(z_1, z_2) = \operatorname{Im} z_2 - F(z_1, \operatorname{Re} z_2)$, where F is a smooth function and $F(0) = 0$. As $\frac{\partial r}{\partial z_2}(\xi) \neq 0$, in a neighborhood of ξ we can define the following vector fields

$$L_1 := \frac{\partial}{\partial z_1} - \left(\frac{\partial r}{\partial z_2}\right)^{-1} \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}, \text{ and } L_2 := \frac{\partial}{\partial z_2}.$$

Note that $L_1 r \equiv 0$ and L_1, L_2 form a basis of $T^{1,0}_z$ for all z near ξ. For any $j, k > 0$, set

$$\mathcal{L}_{j,k}(z) := \underbrace{L_1 \dots L_1}_{j-1 \text{ times}} \underbrace{\bar{L}_1 \dots \bar{L}_1}_{k-1 \text{ times}} \partial\bar{\partial} r(L_1, \bar{L}_1)(z).$$

As $\xi \in \partial\Omega$ is a point of type m_ξ, from the results in [3, Theorem 2.4], it follows that there exist j_0, k_0 with $j_0 + k_0 = m_\xi$ which satisfy

$$\mathcal{L}_{j,k}(\xi) = 0 \quad j + k < m_\xi, \text{ and } \mathcal{L}_{j_0,k_0}(\xi) \neq 0.$$

Denote

$$C^\xi_l(z) = \max\left\{\left|\mathcal{L}_{j,k}(z)\right| : j + k = l\right\}.$$

Let $X = b_1 L_1 + b_2 L_2$ be a holomorphic tangent vector at z. Now define the *Catlin metric*

$$M_\xi(z, X) := \frac{|b_2|}{|r(z)|} + |b_1| \sum_{l=2}^{m_\xi} \left(\frac{C^\xi_l(z)}{|r(z)|}\right)^{\frac{1}{l}}.$$

Owing to the result of Catlin [7, Theorem 1], the Kobayashi metric is locally bi-Lipschitz to the Catlin metric.

Theorem 2.2 *Let $\Omega = \{z \in \mathbb{C}^2 : r(z) < 0\}$ be a smoothly bounded pseudoconvex domain. If $\xi \in \partial\Omega$ be a point of finite type m_ξ, then there exist a neighborhood U of ξ and a constant $C \geq 1$ such that*

$$\frac{1}{C} M_\xi(z, X) \leq K_\Omega(z, X) \leq C M_\xi(z, X)$$

for each $z \in \Omega \cap U$ and $X \in \mathbb{C}^2$.

For our convenience, we may make a small change to the form of Catlin metrics. If X is a holomorphic tangent vector at z, then it has the unique orthogonal decomposition

$X = X_H + X_N$ by $X_H \in H_{\pi(z)}\partial\Omega$ and $X_N \in N_{\pi(z)}\partial\Omega$. Here $\pi(z)$ is the closest point projected to the boundary (see [1, Lemma2.1]). As L_2 may be not parallel to X_N, we set

$$\widetilde{M}_\xi(z, X) := \frac{|X_N|}{\delta(z)} + |X_H| \sum_{l=2}^{m_\xi} \left(\frac{C_l^\xi(z)}{\delta(z)} \right)^{\frac{1}{l}}.$$

Lemma 2.2 in [15] implies that $M_\xi(z, X) \asymp \widetilde{M}_\xi(z, X)$ in a neighborhood U of ξ.

Now choose open neighborhoods U_i of $\xi_i \in \partial\Omega$, $1 \le i \le s$, which form a finite cover of $\partial\Omega$. There exists a small $\varepsilon > 0$ such that the neighborhood

$$N_\varepsilon(\partial\Omega) := \{z \in \Omega : \delta_\Omega(z) < \varepsilon\} \subset \bigcup_{i=1}^{s} U_i.$$

Denote $I_z := \{i : z \in \overline{U}_i\}$, and set

$$\widetilde{M}(z, X) := \max_{i \in I_z} \{\widetilde{M}_{\xi_i}(z, X)\} \text{ for } z \in \Omega \cap N_\varepsilon(\partial\Omega).$$

Since it is upper semi-continuous, we can define a global Finsler metric in Ω by

$$\widetilde{K}(z, X) := K(z, X)S(z, X)$$

with the positive function $S(z, X) \asymp 1$, and $\widetilde{K}(z, X) = \widetilde{M}(z, X)$ for $z \in \Omega \cap N_\varepsilon(\partial\Omega)$, which implies that $\widetilde{K}(z, X) \asymp K(z, X)$.

Denoting $m = \max\{m_{\xi_i}\}$, for $z \in \Omega \cap N_\varepsilon(\partial\Omega)$ and $X \in \mathbb{C}^2$, it now follows that

$$\frac{|X_N|}{\delta(z)} + \frac{1}{C}\frac{|X_H|}{\delta(z)^{\frac{1}{m}}} \le \widetilde{K}(z, X) \le \frac{|X_N|}{\delta(z)} + C\frac{|X_H|}{\delta(z)^{\frac{1}{2}}} \tag{2.1}$$

for some constant $C > 0$. We may call it a Catlin-type metric, and denote by $d_{\widetilde{K}}$ the distance associated to this metric.

By using the proof of [15, Theorem 1.12], we know Catlin-type distances satisfy the following estimates.

Lemma 2.3 *Assume that $\Omega \subset \mathbb{C}^2$ is a smoothly bounded pseudoconvex domain of finite type. Then there exists a constant $C \ge 1$ such that*

$$\left| \log\left(\frac{\delta(x)}{\delta(y)}\right) \right| - C \le d_{\widetilde{K}}(x, y) \le 2\log\left(1 + \frac{|x - y|}{\sqrt{\delta(x)\delta(y)}}\right) + C$$

for each $x, y \in \Omega$.

2.4 Gromov hyperbolicity

In this section we will give some definitions and results about Gromov hyperbolicity. Refer to [5] for further details.

Let (X, d) be a metric space. The *Gromov product* of two points $x, y \in X$ with respect to a base point $\omega \in X$ is defined by

$$(x|y)_\omega := \frac{1}{2}\left(d(x, \omega) + d(y, \omega) - d(x, y)\right).$$

Recall that a metric space (X, d) is a *geodesic* space if any two distinct points $x, y \in X$ can be joined by a *geodesic segment*. Furthermore, a metric space (X, d) is called *proper*

if every closed ball in (X, d) is compact. A proper geodesic metric space X is called *Gromov hyperbolic* if there is a constant $\delta \geq 0$ such that, for any $x, y, z, \omega \in X$,

$$(x|y)_\omega \geq \min\{(x|z)_\omega, (z|y)_\omega\} - \delta. \tag{2.2}$$

An equivalent definition of Gromov hyperbolicity is that each geodesic triangle is δ-thin for some $\delta > 0$, i.e., each side lies in the δ-neighborhood of the other sides.

Moreover, for a Gromov hyperbolic space (X, d), the Gromov product has a classic estimation (see inequality (8.4) in [6]). If γ is a geodesic connecting x and y in X, then there exist a constant $\delta' > 0$ such that

$$|(x|y)_\omega - d(\omega, \gamma)| \leq \delta'. \tag{2.3}$$

Definition 2.4 Let (X, d) be a metric space and $I \subset \mathbb{R}$ be an interval. For $\lambda \geq 1$ and $\kappa \geq 0$, a map $\gamma : I \to X$ is called a (λ, κ)-quasi-geodesic if for all $s, t \in I$,

$$\lambda^{-1}|t - s| - \kappa \leq d(\gamma(t), \gamma(s)) \leq \lambda|t - s| + \kappa.$$

In particular, γ is called a geodesic when $\lambda = 1, \kappa = 0$, and γ is called a λ-quasi-geodesic when $\kappa = 0$.

Remark 2.5 In the proof of Theorem 3.1, we reparamerized the interval I to $[0, 1]$. Then for a Finsler metric space (Ω, d_F), a λ-quasi-geodesic $\gamma : [0, 1] \to \Omega$ satisfies

$$\lambda^{-1} L_F(\gamma|_{[s,t]}) \leq d_F(\gamma(t), \gamma(s)) \leq \lambda L_F(\gamma|_{[s,t]}), \quad \forall s, t \in [0, 1].$$

Theorem 2.6 [5, Part III: Theorem 1.7] *Suppose that (X, d) is a δ-Gromov hyperbolic geodesic space with $\delta > 0$. And suppose that γ is a (λ, ε)-quasi-geodesic in X and $[p, q]$ is a corresponding geodesic segment joining the endpoints of γ. Then there exists a constant $R = R(\delta, \lambda, \varepsilon)$ which satisfies that the Hausdorff distance between $[p, q]$ and the image of γ is less than R.*

3 Estimate of the Finsler distance

In this section we first prove Theorem 1.4. For convenience, we restate it as follows.

Theorem 3.1 *Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a bounded Gromov hyperbolic domain respect to a complete Finsler distance d_F. Suppose that there exist constants $C_1, C_2, \alpha > 0, 1 > \beta > 0$ with*

$$d_F(x, y) \geq \alpha \left| \log \left(\frac{\delta(x)}{\delta(y)} \right) \right| - C_1, \quad \text{for any } x, y \in \Omega \tag{3.1}$$

and the Finsler metric

$$F(z, X) \geq \frac{C_2 |X|}{\delta(z)^\beta}, \quad \text{for any } z \in \Omega, \ 0 \neq X \in \mathbb{C}^n. \tag{3.2}$$

For each $x, y \in \Omega$, let $L(\gamma)$ denote the Euclidean length of a Finsler λ-quasi-geodesic γ connecting x and y in Ω. Then there exists a constant $C = C(\lambda) > 0$ such that

$$d_F(x, y) \geq 2\alpha \log \left(\frac{L(\gamma)^{\frac{1}{\beta}}}{\sqrt{\delta(x)\delta(y)}} \right) - C.$$

Proof As d_F is a complete Finsler distance on Ω, the Hopf-Rinow Theorem (see [5, Part I: Proposition 3.7]) implies that any two points in Ω can be connected by a Finsler geodesic. Now for any λ-quasi-geodesic $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, let $\eta : [0, 1] \to \Omega$ be a geodesic with $\eta(0) = x$ and $\eta(1) = y$. Define $H := \max_{z\in\gamma} \delta(z)$. There exists $t_0 \in [0, 1]$ with $H = \delta(\gamma(t_0))$. Considering the subcurves $\gamma_1 = \gamma|_{[0,t_0]}$ and $\gamma_2 = \gamma|_{[t_0,1]}$, there are two possibilities:

If $H \geq L(\gamma)^{\frac{1}{\beta}}$, as (Ω, d_F) is Gromov hyperbolic with $\delta > 0$, there exists a constant $R = R(\delta, \lambda)$ such that the Hausdorff distance between γ and η is less than R. Choosing $t_0' \in [0, 1]$ such that $d_K(\gamma(t_0), \eta(t_0')) \leq R$, then by (3.1) we have

$$H = \delta(\gamma(t_0)) \asymp \delta(\eta(t_0'))$$

and

$$d_F(x, y) = d_F(x, \eta(t_0')) + d_F(\eta(t_0'), y) \geq \alpha \log\left(\frac{\delta(\eta(t_0'))}{\delta(x)}\right) + \alpha \log\left(\frac{\delta(\eta(t_0'))}{\delta(y)}\right).$$

Thus

$$d_F(x, y) \geq 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) - C \geq 2\alpha \log\left(\frac{L(\gamma)^{\frac{1}{\beta}}}{\sqrt{\delta(x)\delta(y)}}\right) - C,$$

which completes the proof.

The other possibility is $H < L(\gamma)^{\frac{1}{\beta}}$. Since $\delta(x) \leq H$, there exists $k \in \mathbb{N}_+$ with

$$2^{-\frac{k}{\beta}} H < \delta(x) \leq 2^{-\frac{k-1}{\beta}} H.$$

Then we shall consider the following three alternatives:

(a). Consider the curve γ_1 and define $0 = s_0 \leq s_1 < \cdots < s_k \leq t_0$ as follows,

$$s_j = \min\left\{s \in [0, t_0] : \delta(\gamma(s)) = \frac{H}{2^{\frac{k-j}{\beta}}}\right\}, \quad j = 1, \ldots, k.$$

By denoting $x_j = \gamma\left(s_j\right)$, $j = 0, \ldots, k$, we have

$$1 \leq \frac{\delta\left(x_j\right)}{\delta\left(x_{j-1}\right)} \leq 2^{\frac{1}{\beta}}. \tag{3.3}$$

In the first case we assume that there exists an index $l \in \{1, \ldots, k\}$ with

$$L(\gamma|_{[s_{l-1},s_l]}) > \frac{1}{8} 2^{-(k-l)} L(\gamma).$$

Then, for $t \in \left[s_{l-1}, s_l\right]$, we have

$$\delta(\gamma(t)) \leq 2^{-\frac{k-l}{\beta}} H,$$

which implies that

$$\begin{aligned} L_F\left(\gamma|_{[s_{l-1},s_l]}\right) &\gtrsim \int_{s_{l-1}}^{s_l} \frac{|\dot\gamma(t)|}{\delta(\gamma(t))^\beta} dt \geq \frac{2^{k-l}}{H^\beta} \int_{s_{l-1}}^{s_l} |\dot\gamma(t)| dt \\ &= \frac{2^{k-l} L(\gamma|_{[s_{l-1},s_l]})}{H^\beta} \gtrsim \frac{L(\gamma)}{H^\beta}. \end{aligned} \tag{3.4}$$

For $t_1 := s_k \le t_0$, we choose $t_1' \in [0,1]$ such that $d_F(\gamma(t_1), \eta(t_1')) \le R$. Then we have $H = \delta(\gamma(t_1)) \asymp \delta(\eta(t_1'))$. Let $[\gamma(t_1), \eta(t_1')]$ be a Finsler geodesic connecting $\gamma(t_1)$ and $\eta(t_1')$. As (Ω, d_F) is Gromov hyperbolic and $\eta|_{[0,t_1']} \cup [\gamma(t_1), \eta(t_1')] \cup \gamma|_{[0,t_1']}$ is a quasi-geodesic triangle, there exists a constant $M = M(\delta, \lambda)$ such that $x_{l-1}, x_l \in N_M(\eta|_{[0,t_1']} \cup [\gamma(t_1), \eta(t_1')])$. Here $N_M(A) := \{z \in \Omega : d_F(z, A) < M\}$ for a set A.

(1) If $x_{l-1}, x_l \in N_M(\eta|_{[0,t_1']})$, choose $x_{l-1}', x_l' \in \eta|_{[0,t_1']}$ such that $d_F(x_{l-1}, x_{l-1}') < M$ and $d_F(x_l, x_l') < M$. It means $\delta(x_{l-1}) \asymp \delta(x_{l-1}')$ and $\delta(x_l) \asymp \delta(x_l')$. Denote $x_{l-1}' = \eta(s_{l-1}')$ and $x_l' = \eta(s_l')$ for $s_{l-1}', s_l' \in [0, t_1']$. Without loss of generality, we may suppose that $s_{l-1}' \le s_l'$. As

$$d_F(x_{l-1}', x_l') \ge d_F(x_{l-1}, x_l) - 2M \ge \frac{1}{\lambda} L_F\left(\gamma|_{[s_{l-1}, s_l]}\right) - 2M,$$

combining (3.3) we have

$$\begin{aligned} d_F(x, y) &= d_F(x, x_{l-1}') + d_F(x_{l-1}', x_l') + d_F(x_l', \eta(t_1')) + d_F(\eta(t_1'), y) \\ &\ge 2\alpha \log\left(\frac{\delta(\eta(t_1'))}{\sqrt{\delta(x)\delta(y)}}\right) + \alpha \log\left(\frac{\delta(x_{l-1}')}{\delta(x_l')}\right) + \frac{1}{\lambda} L_F\left(\gamma|_{[s_{l-1}, s_l]}\right) - 2M \\ &\ge 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) + \widetilde{C}_1 \frac{L(\gamma)}{H^\beta} - \widetilde{C}_2. \end{aligned}$$

(2) If $x_{l-1}, x_l \in N_M([\gamma(t_1), \eta(t_1')])$, we choose $x_{l-1}', x_l' \in [\gamma(t_1), \eta(t_1')]$ such that $d_F(x_{l-1}, x_{l-1}') < M$ and $d_F(x_l, x_l') < M$. So

$$d_F(x_{l-1}, x_l) \le d_F(x_{l-1}, x_{l-1}') + d_F(x_{l-1}', x_l') + d_F(x_l', x_l) < 2M + R.$$

Then

$$\begin{aligned} d_F(x, y) &\ge d_F(x, \eta(t_1')) + d_F(\eta(t_1'), y) + d_F(x_{l-1}, x_l) - 2M - R \\ &\ge 2\alpha \log\left(\frac{\delta(\eta(t_1'))}{\sqrt{\delta(x)\delta(y)}}\right) + \frac{1}{\lambda} L_F\left(\gamma|_{[s_{l-1}, s_l]}\right) - C \\ &\ge 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) + \widetilde{C}_1 \frac{L(\gamma)}{H^\beta} - \widetilde{C}_2. \end{aligned}$$

(3) If x_{l-1}, x_l do not satisfy the previous two cases, without loss of generality, we may suppose that $x_{l-1} \in N_M(\eta|_{[0,t_1']})$ and $x_l \in N_M([\gamma(t_1), \eta(t_1')])$. Choosing $x_{l-1}' \in \eta|_{[0,t_1']}$ and $x_l' \in [\gamma(t_1), \eta(t_1')]$ such that $d_F(x_{l-1}, x_{l-1}') < M$ and $d_F(x_l, x_l') < M$, we have

$$\begin{aligned} d_F(x_{l-1}, x_l) &\le d_F(x_{l-1}, x_{l-1}') + d_F(x_{l-1}', \eta(t_1')) + d_F(\eta(t_1'), x_l') + d_F(x_l', x_l) \\ &< d_F(x_{l-1}', \eta(t_1')) + 2M + R. \end{aligned}$$

Then we have

$$\begin{aligned} d_F(x, y) &\ge d_F(x, x_{l-1}') + d_F(x_{l-1}', \eta(t_1')) + d_F(\eta(t_1'), y) + d_F(\eta(t_1'), x_l') - R \\ &\ge 2\alpha \log\left(\frac{\delta(\eta(t_1'))}{\sqrt{\delta(x)\delta(y)}}\right) + \alpha \log\left(\frac{\delta(x_{l-1}')}{\delta(x_l')}\right) + d_F(x_{l-1}, x_l) - 2M - 2R \\ &\ge 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) + \widetilde{C}_1 \frac{L(\gamma)}{H^\beta} - \widetilde{C}_2. \end{aligned}$$

(b). Now consider the curve γ_2 and define $1 = s_0^* \geq s_1^* > \cdots > s_k^* \geq t_0$ as follows,

$$s_j^* = \max\left\{ s \in [t_0, 1] : \delta(\gamma(s)) = \frac{H}{2^{\frac{k-j}{\beta}}} \right\}, \quad j = 1, \ldots, k^*.$$

By denoting $x_j^* = \gamma\left(s_j^*\right)$, $j = 0, \ldots, k^*$, we have

$$1 \leq \frac{\delta\left(x_j^*\right)}{\delta\left(x_{j-1}^*\right)} \leq 2^{\frac{1}{\beta}}.$$

The second alternative is that there exists an index $l \in \{1, \ldots, k^*\}$ with

$$L(\gamma|_{[s_l^*, s_{l-1}^*]}) > \frac{1}{8} 2^{-(k^*-l)} L(\gamma).$$

Denote $t_2 := s_k^* \geq t_0$. By applying the same considerations to the curve γ_2 instead of γ_1, we can see that

$$d_F(x, y) \geq 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) + \widetilde{C}_1 \frac{L(\gamma)}{H^\beta} - \widetilde{C}_2.$$

(c). The third alternative is

$$L(\gamma|_{[s_{j-1}, s_j]}) \leq \frac{1}{8} 2^{-(k-j)} L(\gamma), \quad j = 1, \ldots, k,$$

and

$$L(\gamma|_{[s_{j-1}^*, s_j^*]}) \leq \frac{1}{8} 2^{-(k^*-j)} L(\gamma), \quad j = 1, \ldots, k^*.$$

Then

$$L(\gamma|_{[0,t_1]}) = \sum_{j=1}^{k} L(\gamma|_{[s_{j-1}, s_j]}) \leq \frac{1}{4} L(\gamma) \text{ and } L(\gamma|_{[t_2,1]}) = \sum_{j=1}^{k^*} L(\gamma|_{[s_{j-1}^*, s_j^*]}) \leq \frac{1}{4} L(\gamma).$$

follows. We have

$$L(\gamma|_{[t_1,t_2]}) = L(\gamma) - L(\gamma|_{[0,t_1]}) - L(\gamma|_{[t_2,1]}) \geq \frac{1}{2} L(\gamma).$$

Then similar to inequality (3.4), it follows that $d_F(\gamma(t_1), \gamma(t_2)) \gtrsim \frac{L(\gamma)}{H^\beta}$.

Now we choose $t_1', t_2' \in [0, 1]$ such that $d_F(\gamma(t_1), \eta(t_1')) < R$ and $d_F(\gamma(t_2), \eta(t_2')) < R$. It means that

$$H = \delta(\gamma(t_1)) \asymp \delta(\eta(t_1')), \quad H = \delta(\gamma(t_2)) \asymp \delta(\eta(t_2')).$$

Without loss of generality, we may suppose that $t_1' \leq t_2'$. Therefore, we now obtain

$$\begin{aligned} d_F(x, y) &\geq d_F(x, \eta(t_1')) + d_F(\eta(t_1'), \eta(t_2')) + d_F(\eta(t_2'), y) \\ &\geq 2\alpha \log\left(\frac{\sqrt{\delta(\eta(t_1'))\delta(\eta(t_2'))}}{\sqrt{\delta(x)\delta(y)}}\right) + d_F(\gamma(t_1), \gamma(t_2)) - 2R \\ &\geq 2\alpha \log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) + \widetilde{C}_1 \frac{L(\gamma)}{H^\beta} - \widetilde{C}_2, \end{aligned}$$

which implies that the above estimate is true in any case.

Let

$$f(t) := 2\alpha \log\left(\frac{t}{\sqrt{\delta(x)\delta(y)}}\right) + C\frac{L(\gamma)}{t^{\beta}}.$$

Through a simple calculation, we know the function f gets its minimum value when

$$t = \left(\frac{\beta C}{2\alpha}\right)^{\frac{1}{\beta}} L(\gamma)^{\frac{1}{\beta}}$$

which gives the lower bound

$$d_F(x, y) \geq 2\alpha \log\left(\frac{L(\gamma)^{\frac{1}{\beta}}}{\sqrt{\delta(x)\delta(y)}}\right) - C,$$

which completes the proof. □

Here we would like to provide an additional estimate which can also yield the above result by considering the condition (3.1). Although the index is not more precise by using this approach and it has no direct connection with other aspects, we include it for readers who are interested.

Proposition 3.2 *Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a bounded domain and let d_F be a complete Finsler distance. Suppose that there exist constants $C_1, C_2, \alpha > 0, 1 > \beta > 0$ with*

$$d_F(z, z_0) \leq \alpha \log\left(\frac{C_1}{\delta(z)}\right), \quad \text{for any } z \text{ and a fix } z_0 \in \Omega,$$

and the Finsler metric

$$F(z, X) \geq \frac{C_2\,|X|}{\delta(z)^{\beta}}, \quad \text{for any } z \in \Omega,\ 0 \neq X \in \mathbb{C}^n.$$

For each $x, y \in \Omega$, let $L(\gamma)$ denote the Euclidean length of a Finsler λ-quasi-geodesic γ connecting x and y in Ω and $H_\gamma := \max_{z\in\gamma} \delta(z)$. Then there exists a constant $C = C(\lambda) > 0$ such that

$$L(\gamma) < C \text{ and } H_\gamma^{\beta} \gtrsim \frac{L(\gamma)}{\log \frac{2C}{L(\gamma)}}.$$

Proof For a Finsler λ-quasi-geodesic $\gamma : [a, b] \to \Omega$ with $\gamma(a) = x$ and $\gamma(b) = y$, there exists $T \in [a, b]$ such that $H_\gamma = \delta(\gamma(T))$. Since

$$\begin{aligned}\frac{1}{\lambda}|T - t| \leq d_F(\gamma(T), \gamma(t)) &\leq d_F(\gamma(T), z_0) + d_F(z_0, \gamma(t)) \\ &\leq \alpha \log\left(\frac{1}{\delta(\gamma(T))\delta(\gamma(t))}\right) + C,\end{aligned}$$

we have

$$\delta(\gamma(t)) \leq \sqrt{\delta(\gamma(T))\delta(\gamma(t))} \leq \exp\left(-\frac{|T - t|}{2\lambda\alpha} + C\right).$$

As γ is a λ-quasi-geodesic, it follows that $F(\gamma(t), \dot{\gamma}(t)) \leq \lambda$ almost everywhere. Then

$$\begin{aligned} L(\gamma) &= \int_a^b |\dot{\gamma}(t)|dt \leq \frac{\lambda}{C_2}\int_a^b \delta(\gamma(t))^\beta dt \\ &\lesssim \int_{[a,b]\cap[T-M,T+M]} \delta(\gamma(t))^\beta dt + \int_{[a,b]\cap[T-M,T+M]^c} \exp\left(-\frac{\beta|T-t|}{2\lambda\alpha} + C\right) dt \\ &\lesssim 2MH_\gamma^\beta + \frac{4\lambda\alpha}{\beta}\exp\left(-\frac{\beta M}{2\lambda\alpha} + C\right) \leq CMH_\gamma^\beta + C\exp\left(-\frac{\beta M}{2\lambda\alpha}\right). \end{aligned}$$

As H_γ has an upper bound in Ω, there exist a constant $\widetilde{C} > 0$ such that $L(\gamma) < \widetilde{C}$. Choosing $C > \widetilde{C}$ and letting $M = \frac{2\lambda\alpha}{\beta}\log\frac{2C}{L(\gamma)} > 0$, we have $C\exp\left(-\frac{\beta M}{2\lambda\alpha}\right) = \frac{L(\gamma)}{2}$. Hence

$$H_\gamma^\beta \geq \frac{L(\gamma)}{2CM} \gtrsim \frac{L(\gamma)}{\log\frac{2C}{L(\gamma)}},$$

which completes the proof. □

Remark 3.3 Note that in [16, Lemma 3.1], Liu-Wang-Zhou obtained similar estimates for m-convex domains. Later, Nikolov-Ökten [18, Corollary 11] generalized the result to strongly Goldilocks domains as defined by them. However, previous proofs are based on the division of curves. Here we give a proof in integral form.

Now we begin to prove the Gehring-Hayman type theorem for smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$.

Theorem 3.4 *Let $\Omega \subset \mathbb{C}^2$ be is a smoothly bounded pseudoconvex domain of finite type m. Then there exists a constant $C = C(\lambda) > 0$ such that for every $x, y \in \Omega$*

$$L(\gamma) \leq C|x-y|^{\frac{1}{m}},$$

where γ is a Kobayashi λ-quasi-geodesic connecting x and y with $\lambda \geq 1$. Moreover, denoting $H_\gamma = \max_{z\in\gamma}\delta(z)$, there exists a constant $C' = C'(\lambda) > 0$ such that

$$H_\gamma^{\frac{1}{m}} \geq C'L(\gamma).$$

Proof Since Kobayashi distance is bi-Lipschitz to Catlin-type distance, the Kobayashi λ-quasi-geodesic γ is Catlin-type λ'-quasi-geodesic. By Theorem 1.4 and Lemma 2.3, from inequality (2.1) we have

$$2\log\left(\frac{L(\gamma)^m}{\sqrt{\delta(x)\delta(y)}}\right) - C \leq d_{\widetilde{K}}(x,y) \leq 2\log\left(1 + \frac{|x-y|}{\sqrt{\delta(x)\delta(y)}}\right) + C. \tag{3.5}$$

It follows that

$$L(\gamma)^m \lesssim |x-y| + \sqrt{\delta(x)\delta(y)}.$$

Hence if $\sqrt{\delta(x)\delta(y)} \leq |x-y|$, we have the desired estimation $L(\gamma) \lesssim |x-y|^{\frac{1}{m}}$.

When $\sqrt{\delta(x)\delta(y)} > |x-y|$, let η be a Catlin-type geodesic connecting x and y. Denoting $H := \max_{z\in\eta}\delta(z)$, it follows from Lemma 2.3 that

$$2\log\left(\frac{H}{\sqrt{\delta(x)\delta(y)}}\right) \leq d_{\widetilde{K}}(x,y) \leq 2\log\left(1 + \frac{|x-y|}{\sqrt{\delta(x)\delta(y)}}\right) + C,$$

which implies $H \lesssim |x-y| + \sqrt{\delta(x)\delta(y)}$.

From the result of Fiacchi [8] or Li-Pu-Wang [14] recently, we know $(\Omega, d_{\widetilde{K}})$ is Gromov hyperbolic with $\delta > 0$. Applying Theorem 2.6, there exists a constant $R = R(\delta, \lambda')$ such that the Hausdorff distance between η and γ is less than R. Denoting $H_\gamma := \max_{\omega\in\gamma} \delta(\omega) = \delta(\omega_0)$, then there exists a point $z_0 \in \eta$ with $d_{\widetilde{K}}(\omega_0, z_0) \leq R$.

Applying Lemma 2.3, we have $\delta(\omega_0) \asymp \delta(z_0)$, which implies that

$$H_\gamma \lesssim \delta(z_0) \leq H \lesssim |x-y| + \sqrt{\delta(x)\delta(y)}.$$

By using Lemma 2.1, we get

$$\begin{aligned}\frac{L(\gamma)}{H_\gamma^{\frac{1}{m}}} &\leq \int_0^1 \frac{|\dot\gamma(t)|}{\delta(\gamma(t))^{\frac{1}{m}}} dt \lesssim \int_0^1 K(\gamma(t), \dot\gamma(t))dt = L_K(\gamma) \leq \lambda d_K(x,y)\\ &\lesssim \log\left(1 + \frac{C|x-y|}{\sqrt{\delta(x)\delta(y)}}\right) \lesssim \frac{|x-y|}{\sqrt{\delta(x)\delta(y)}}.\end{aligned}$$

Then

$$L(\gamma) \lesssim \frac{\left(|x-y| + \sqrt{\delta(x)\delta(y)}\right)^{\frac{1}{m}} |x-y|}{\sqrt{\delta(x)\delta(y)}} \lesssim |x-y|^{\frac{1}{m}}.$$

Similarly, from the Gromov hyperbolicity, it also has $H \lesssim H_\gamma$. Fix a point $w \in \Omega$. By using inequalities (2.3) and (3.5), we obtain that

$$\log\left(\frac{\delta(w)}{H}\right) - C \leq d_{\widetilde{K}}(w, \eta) \leq (x|y)_w^{\widetilde{K}} + \delta' \leq \log\left(\frac{C}{L(\gamma)^m}\right).$$

Hence $H_\gamma \gtrsim H \gtrsim L(\gamma)^m$, which completes the proof. □

4 The proof of Theorem 1.5

To prove Theorem 1.5, we first require the localization result of Kobayashi metric. This is based on Royden's Localization Lemma [20, Lemma 2], with its proof available in [11, Lemma 4]. Noting $\tanh(x) \geq 1 - 2e^{-2x}$ for $x \geq 0$, the Lemma 3.1 in [21] can be stated as follows.

Lemma 4.1 *Let $\Omega \subset \mathbb{C}^n$ is a Kobayashi hyperbolic domain and U is an open subset of $\mathbb{C}^n$ such that $U \cap \Omega \neq \emptyset$ and connected. Then for every $W \subset\subset U$ with $W \cap \Omega \neq \emptyset$ and $d_K(W \cap \Omega, \Omega\backslash U) > 0$, there exists a constant $L > 0$ such that*

$$K_\Omega(z, X) \leq K_{U\cap\Omega}(z, X) \leq \left(1 + Le^{-2d_K(z,\Omega\backslash U)}\right) K_\Omega(z, X).$$

for all $z \in W \cap \Omega$ and $X \in \mathbb{C}^n$.

Proof of Theorem 1.5 By Lemma 4.1, it follows that there exist a neighborhood V_0 of ξ with $V_0 \subset\subset U$ and a constant $C > 0$ such that, for $z \in \Omega \cap V_0$ and $X \in \mathbb{C}^n$,

$$K_\Omega(z, X) \leq K_{\Omega\cap U}(z, X) \leq \left(1 + Ce^{-2d_K(z,\Omega\backslash U)}\right) K_\Omega(z, X).$$

Hence we only need to check the right side of inequality (1.3) and (1.4). By using the lower bound in [15, Theorem 1.12], we have

$$d_{\widetilde{K}}(z, \Omega\backslash U) \geq \min_{\omega\in\Omega\backslash U} 2\log\left(\frac{|z-\omega|^m + \delta(z) \vee \delta(\omega)}{\sqrt{\delta(z)\delta(\omega)}}\right) - C \geq \log\frac{1}{\delta(z)} - C.$$

Then there exists a constant $A \geq 1$ such that

$$d_K(z, \Omega\backslash U) \geq \frac{1}{A} d_{\widetilde{K}}(z, \Omega\backslash U) \geq \frac{1}{A} \log \frac{1}{\delta(z)} - C,$$

which means that

$$K_{\Omega\cap U}(z, X) \leq \left(1 + C\delta(z)^{\frac{2}{A}}\right) K_\Omega(z, X). \tag{4.1}$$

For a Kobayashi geodesic $\gamma : [a, b] \to \Omega$ with $\gamma(a) = x$ and $\gamma(b) = y$, there exists $T \in [a, b]$ such that $H_\gamma = \delta(\gamma(T)) = \max_{t\in[a,b]} \delta(\gamma(t))$. By using the proof of Theorem 1.2, it follows that

$$H_\gamma \lesssim |x - y| + \sqrt{\delta(x)\delta(y)} \text{ and } L(\gamma) \lesssim |x - y|^{\frac{1}{m}}.$$

As $L(\gamma) \to 0$ when $|x - y| \to 0$, we can choose a neighborhood V of ξ with $V \subset V_0$ such that, for any $x, y \in \Omega \cap V$, the Kobayashi geodesic γ connecting x and y is in $\Omega \cap V_0$. Since

$$\begin{aligned} |T - t| = d_K(\gamma(T), \gamma(t)) &\leq \log\left(1 + \frac{C|x - y|}{\sqrt{\delta(\gamma(T))\delta(\gamma(t))}}\right) \\ &\leq \log\left(\frac{H_\gamma + L(\gamma)}{\sqrt{\delta(\gamma(T))\delta(\gamma(t))}}\right) + C, \end{aligned}$$

we have

$$\delta(\gamma(t)) \leq \sqrt{\delta(\gamma(T))\delta(\gamma(t))} \leq \left(H_\gamma + L(\gamma)\right) e^{-|T-t|}.$$

Then for $x, y \in \Omega \cap V$, when $|x - y| \geq \sqrt{\delta(x)\delta(y)}$ we obtain that

$$\int_a^b \delta(\gamma(t))^{\frac{2}{A}} K_\Omega(\gamma(t), \dot\gamma(t))dt \lesssim \left(H_\gamma + L(\gamma)\right)^{\frac{2}{A}} \int_{\mathbb{R}} e^{-|T-t|}dt \lesssim |x - y|^{\frac{2}{Am}}.$$

And when $|x - y| < \sqrt{\delta(x)\delta(y)}$, we have

$$\begin{aligned} \int_a^b \delta(\gamma(t))^{\frac{2}{A}} K_\Omega(\gamma(t), \dot\gamma(t))dt &\lesssim H_\gamma^{\frac{2}{A}} L_K(\gamma) \lesssim H_\gamma^{\frac{2}{A}} \log\left(1 + \frac{C|x - y|}{\sqrt{\delta(x)\delta(y)}}\right) \\ &\lesssim \frac{\left(|x - y| + \sqrt{\delta(x)\delta(y)}\right)^{\frac{2}{A}} |x - y|}{\sqrt{\delta(x)\delta(y)}} \lesssim |x - y|^{\frac{2}{A}}. \end{aligned}$$

Hence

$$d_{K_{\Omega\cap U}}(x, y) \leq L_K(\gamma) + C \int_a^b \delta(\gamma(t))^{\frac{2}{A}} K_\Omega(\gamma(t), \dot\gamma(t))dt \leq d_{K_\Omega}(x, y) + C|x - y|^{\frac{2}{Am}}.$$

Additionally, for $z \in \gamma$ it follows that

$$\delta_\Omega(z) \leq \delta_\Omega(x) \wedge \delta_\Omega(y) + L(\gamma) \lesssim \delta_\Omega(x) \wedge \delta_\Omega(y) + |x - y|^{\frac{1}{m}}.$$

Therefore, for $x \neq y \in \Omega \cap V$, by estimate (4.1) we deduce that

$$\frac{d_{K_{\Omega\cap U}}(x, y)}{d_{K_\Omega}(x, y)} \leq 1 + C\left(\delta_\Omega(x) \wedge \delta_\Omega(y) + |x - y|^{\frac{1}{m}}\right)^{\frac{2}{A}},$$

which completes the proof. □

Acknowledgements The authors would like to thank Professor Jinsong Liu for many precious suggestions. We would also like to thank the referee for a careful reading and valuable comments.

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Check for updates

Received: 4 August 2022 | Accepted: 28 July 2023

DOI: 10.1002/mana.202200325

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Carleson measures on convex domains with smooth boundary of finite type

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Funding information
National Key R&D Program of China, Grant/Award Number: 2021YFA1003100; Key Research Program of Frontier Sciences, CAS, Grant/Award Number: ZDBS-LY-7002; Young Scientist Program of the Ministry of Science and Technology of China, Grant/Award Number: 2021YFA1002200; National Natural Science Foundation of China, Grant/Award Numbers: 11901205, 11925107, 12071155

Abstract

Following M. Abate and A. Saracco's work on strongly pseudoconvex domains in $\mathbb{C}^n$, we characterize Carleson measures of $A^2(D)$ in bounded convex domains in $\mathbb{C}^n$ with smooth boundary of finite type. We also give examples of Carleson measures with uniformly discrete (with respect to the Kobayashi distance) sequences.

KEYWORDS

Bergman spaces, Carleson measures

1 | INTRODUCTION

Let A be a Banach space of holomorphic functions on a domain $D \subset \mathbb{C}^n$. Assume that A is contained in $L^p(D)$ for some $p > 0$. Recall that a finite positive Borel measure μ on D is a Carleson measure of A if there exists a constant $C > 0$ such that

$$\int_D |f|^p d\mu \leqslant C\|f\|_p^p, \qquad \forall f \in A.$$

Given $1 \leq p < +\infty$, the Bergman space $A^p(D)$ of D is the Banach space of holomorphic L^p-functions on D. That is, $A^p(D) = L^p(D) \cap \mathcal{O}(D)$, endowed with the L^p-norm.

Carleson measures for Bergman spaces were first described by Hastings [9], and independently by Oleinik [17] and Pavlov–Oleinik [16]. Subsequently, Cima–Wogen [7] adapted the result to the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ by using spherical caps at the boundary. Then, as essentially noticed by Luecking [11] and explicitly stated by Duren–Weir [8], it is possible to give a characterization for the Carleson measures of Bergman spaces of $\mathbb{B}^n$ by using the balls for the Bergman (or Kobayashi, or pseudohyperbolic) distance. In 1995, Cima–Mercer [6] characterized Carleson measures of the Bergman spaces of strongly pseudoconvex domains. In particular, they proved that the class of Carleson measures of $A^p(D)$ does not depend on p.

A particularly important Bergman space is, of course, $A^2(D)$, where the Bergman kernel lives. This suggests the question of whether it is possible to characterize Carleson measures using the Bergman kernel. This has been done by Duren–Weir [8] for the unit ball, and by Abate–Saracco [4] for the strongly pseudoconvex domains. After that, Abate–Raissy–Sarroco [3], Hu–Lv–Zhu [10], Abate–Raissy [2], and Abate–Mongodi–Raissy [1] obtained a series of results about Bergman space on strongly pseudoconvex domains. Our first main result is a similar characterization of Carleson measures in convex domains with a smooth boundary of finite type.

Let $K : D \times D \to \mathbb{C}$ be the Bergman kernel of a bounded domain $D \subset \mathbb{C}^n$ For any finite positive Borel measure μ on D, the Berezin transform of μ is the function $B\mu : D \to \mathbb{R}$ given by

$$B\mu(z) = \int_D \frac{|K(\zeta, z)|^2}{K(z, z)} d\mu(\zeta).$$

Now, we can prove the promised characterization for the Carleson measures of $A^2(D)$. We recall that ν denotes the Lebesgue volume measure of $\mathbb{R}^{2n}$, normalized so that $\nu(B_1(0)) = 1$. We show (see Theorem 3.1) the following:

Theorem 1.1. *Let μ be a finite positive Borel measure on a bounded convex domain $D \subset \mathbb{C}^n$ with smooth boundary of finite type. Then, the following statements are equivalent:*

(1) μ is a Carleson measure of $A^2(D)$;
(2) the Berezin transform of μ is bounded;
(3) there exists $r_0 \in (0, 1)$, such that for any $r \in (0, r_0)$ and $z \in D$, we have

$$\mu(B_D(z, r)) \leqslant C_r \nu(B_D(z, r))$$

for some $C_r > 0$.

To prove the above theorem, we need the construction of the polydisk and estimates of the Bergman kernel in convex domains of finite type due to Chen [5] and McNeal [13] (needs a little correction, see [14]), which is mainly used in the proof related to (3). With these estimates of the Kobayashi metric and the Bergman kernel on the convex domains with smooth boundary of finite type, we can repeat most technical lemmas in [4] (Lemmas 2.2, 2.6, and 2.7). To prove Lemma 3.2 and Corollary 3.3 in [4] (Lemma 2.11 and Corollary 2.12 in this paper), Abate and Saracco used the Fefferman's result on the estimates of the Bergman kernel on strongly pseudoconvex domains, which may not hold on the convex domains with boundary of finite type. However, we find it can be easily replaced by an estimate of the derivatives of the Bergman kernel and the mean value theorem. Then, the proof of Theorem 1.1 is just the same as Theorem 3.4 in [4].

Next, we are going to construct explicit examples of Carleson measures in convex domains with finite-type boundary. As in the unit disc and the unit ball, an important family of examples is provided by uniformly discrete sequences. Let (X, d) be a metric space. Then, a sequence of points $\{x_j\} \subset X$ is uniformly discrete if there exists $\delta > 0$ such that $d(x_j, x_k) \geqslant \delta$ for all $j \neq k$.

The following result is similar with the results proved about the unit ball [8, 12] and the strongly pseudoconvex domains [4].

Theorem 1.2. *Let $D \subset\subset \mathbb{C}^n$ be a convex bounded domain with finite-type boundary, and let $\Gamma = \{z_j\}$ be a sequence in D. Then, Γ is a finite union of uniformly discrete sequences (with respect to the Kobayashi distance) if and only if $\sum_{z_j \in \Gamma} \prod_{i=1}^n \sigma_i^2(z_j) \delta_{z_j}$ is a Carleson measure of $A^2(D)$, where δ_{z_j} is the Dirac measure at z_j and $\sigma_i(z_j)$ is the polyradii defined in Section 2.*

2 | PRELIMINARIES

2.1 | Notation

(1) For $z \in \mathbb{C}^n$, let $|\cdot|$ denote the standard Euclidean norm, and let $|z_1 - z_2|$ denote the standard Euclidean distance of $z_1, z_2 \in \mathbb{C}^n$.

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(2) Given an open set $D \subsetneq \mathbb{C}^n,\ x \in D$, denote

$$\delta_D(x) = \inf\{|x - \xi| : \xi \in \partial D\}.$$

(3) For any curve $\sigma : I \to \mathbb{C}^n$, we denote its Euclidean length by $l_d(\sigma)$ and the Kobayashi length by $l_k(\sigma)$.
(4) Denote by ν the Lebesgue measure.
(5) For all real numbers a, b, we denote $a \vee b := \max\{a,\ b\}$ and $a \wedge b := \min\{a,\ b\}$.

2.2 | Convex domains with boundary of finite type

A point $p \in \partial D$ is of finite type (in the sense of D'Angelo) means that the maximum order of contact of one-dimensional complex analytic varieties with ∂D at p is bounded.

For a neighborhood U of p, let r be the defining function such that $D \cap U = \{z \in U : r(z) < 0\}$. By a rotation of the canonical coordinates, we can arrange that the normal direction to ∂D at p is given by the $\Re z_1$-axis. Then, by using the implicit function theorem, we obtain a local defining function of the form $r(z_1, \dots, z_n) = \Re z_1 - F(\Im z_1, \dots, \Re z_n, \Im z_n)$, where F is a convex function. For $q \in U$ and $\epsilon > 0$, we will consider the level sets

$$\partial D_{q,\epsilon} = \{z \in U | r(z) = \epsilon + r(q)\},$$

which are also convex by the choice of r.

Then, there exists a small ball $V \subset U$ centered at p such that for every $q \in V$ and a sufficiently small $\epsilon > 0$, we can assign coordinates $(z_1, \dots, z_n)$, $z_i = x_i + ix_{n+i}$, centered at q, obtained by translating and rotating the canonical coordinates, and numbers $\tau^i(q, \epsilon)$, which measure the distance from q to $\partial D_{q,\epsilon}$ along the complex line determined by the z_i-axis.

First, choose z_1 so that $dist\left(q,\ \partial D_{q,\epsilon}\right)$ is achieved along the positive x_1-axis. Let $q_{1,\epsilon}$ be the point in $\partial D_{q,\epsilon}$ such that $\tau^1(q, \epsilon) = |q - q_{1,\epsilon}| = dist\left(q, \partial D_{q,\epsilon}\right)$, and let e_1 be the unit vector in the direction of x_1-axis. Next, choose a unit vector e_2 in the orthogonal complement of the space e_1 (the complex linear span of e_1) such that the minimum distance from q to $\partial D_{q,\epsilon}$ along directions orthogonal to e_1 is achieved along the line given by e_2 in a point $q_{2,\epsilon}$. Choose z_2 such that x_2-axis lies in the direction of e_2 and $\tau^2(q, \epsilon) = |q - q_{2,\epsilon}|$. Now continue by choosing a unit vector e_3 in the orthogonal complement of e_1, e_2, until the basis is complete. Also note that the remaining points $z_i,\ i = 3, \dots, n$, have the property that the distance from q to $\partial D_{q,\epsilon}$ within the z_i-axis is achieved on the positive x_i-axis.

Therefore,

$$P(q,\ \epsilon) = \{|z_i| \le \tau_i(q, \epsilon),\ i = 1, \dots, n\}$$

is the corresponding polydisk constructed in terms of the minimal basis in $D \cap U$.

Suppose D is a convex domain with a smooth boundary of finite type. Then, by the compactness of $\partial\Omega$, we can choose $\{(p_j, V_j, U_j),\ j = 1 \cdots K\}$ such that

$$\bigcup_{j=1}^{K} V_j \supset \partial D,$$

where $V_j \subset U_j$ are both open neighborhood of boundary points p_j as above. Take an open neighborhood W of ∂D such that

$$\bigcup_{j=1}^{K} V_j \supset W \supset \partial D. \tag{1}$$

Then, for any $z_0 \in W$, there exists at least one $j \in \mathbb{N}$ such that $z_0 \in V_j$. Denote by $P^j(z_0, \epsilon)$ the polydisk constructed in terms of the minimal basis in $D \cap U_j$.

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Supposing $z_0 \in V_j \cap V_k$ for $j \neq k$, by taking $\epsilon = |r(z_0)|$, we have

$$\tau_i^j(z_0,\ \epsilon) = \tau_i^k(z_0,\ \epsilon) \tag{2}$$

for $i = 1, \dots, n$.

By repeating the above construction, we can also get a global minimal basis in D. For any $q \in D$, choose $q_1 \in \partial D$ such that $\sigma_1(q) := |q - q_1| = \delta_D(q)$. Put $H_1 = q + \operatorname{span}(q_1 - q)^{\perp}$ and $D_1 = D \cap H_1$.

Let $q_2 \in \partial D_1$ with $\sigma_2(q) := |q_2 - q| = \delta_{D_1}(q)$. Put $H_2 = q + \operatorname{span}(q_1 - q,\ q_2 - q)^{\perp}$, $D_2 = D \cap H_2$, and so on. Thus, we get an orthonormal basis consisting of the vectors $e_i = \frac{q_i - q}{\|q_i - q\|}$. Finally, we choose x_i-axis in the direction of $e_i, 1 \leq i \leq n$.

Note that, since D has smooth boundary, the numbers $\sigma_i(q)$ are uniquely determined when q is near ∂D. Actually by (2), we can assume that, for any $z_0 \in W$,

$$\sigma_i(z_0) = \tau_i^j(z_0, \epsilon) \tag{3}$$

for some $j = 1, \dots, K$.

2.3 | The Kobayashi metric

Given a domain $D \subset \mathbb{C}^n$ $(n \geq 2)$, the (infinitesimal) Kobayashi metric is the pseudo-Finsler metric defined by

$$k_D(x; v) = \inf\{|\xi| : f \in \operatorname{Hol}(\mathbb{D}, D),\ \text{ with } f(0) = x, d(f)_0(\xi) = v\}.$$

Define the Kobayashi length of any curve $\sigma : [a, b] \to D$ to be

$$l_k(\sigma) = \int_a^b k_D\big(\sigma(t); \sigma'(t)\big)dt.$$

It is a consequence of a result due to Venturini [19], which is based on an observation by Royden [18], that the Kobayashi pseudo-distance can be given by:

$$d_K(x, y) = \inf_{\sigma}\big\{l_k(\sigma) |\ \sigma : [a, b] \to D \text{ absolutely continuous curve}$$
$$\text{with } \sigma(a) = x \text{ and } \sigma(b) = y\big\}.$$

The main property of the Kobayashi pseudo-distance is that it is contracted by holomorphic maps: If $f : X \to Y$ is a holomorphic map, then

$$\forall z, w \in X \quad d_K^Y(f(z), f(w)) \leqslant d_K^X(z, w).$$

In particular, the Kobayashi distance is invariant under biholomorphisms, and decreases under inclusions: If $D_1 \subset D_2 \subset\subset \mathbb{C}^n$ are two bounded domains, then we have $d_K^{D_2}(z, w) \leqslant d_K^{D_1}(z, w)$ for all $z, w \in D_1$

If X is a hyperbolic manifold, $z_0 \in X$ and $r \in (0, 1)$, then we shall denote by $B_X(z_0, r)$ the Kobayashi ball of center z_0 and radius $(1/2)\log(1 + r)/(1 - r)$. That is,

$$B_X(z_0, r) = \big\{z \in X \mid \tanh d_K^X(z_0, z) < r\big\}.$$

Note that $\rho_X = \tanh K_X$ is still a distance on X because tanh is a strictly convex function on $\mathbb{R}^+$.

There are some estimates concerning the Kobayashi metric on convex domains.

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Lemma 2.1. *Let $D \subset\subset \mathbb{C}^n$ be a convex domain with smooth boundary. Fix $\omega_0 \in \Omega$, there exist $C_1, C_2 > 0$ such that, for any $z \in D$,*

$$C_1 - \frac{1}{2}\log \delta_D(z) \leqslant d_K(\omega_0, z) \leqslant C_2 - \frac{1}{2}\log \delta_D(z). \tag{4}$$

Remark 2.2. It is well known that the bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary of finite type has many similar properties with convex domains in $\mathbb{C}^n$ with smooth boundary of finite type. But we need the coefficients in the lower bound of the Kobayashi metric to be $\frac{1}{2}$, which may fail in the case of pseudoconvex domain in $\mathbb{C}^2$.

Lemma 2.3. *Let $D \subset\subset \mathbb{C}^n$ be a convex domain with smooth boundary of finite type. Then, there exists $C_3 > 0$ such that, for every $z_0 \in D$ and $r \in (0, 1)$, we have the estimate*

$$\frac{C_3}{1-r}\delta_D(z_0) \geq \delta_D(z) \geq \frac{1-r}{C_3}\delta_D(z_0),$$

for any $z \in B_D(z_0, r)$.

Proof. Fix $\omega_0 \in D$. Then, we obtain

$$\begin{aligned} C_1 - \frac{1}{2}\log \delta_D(z) &\leqslant K_D(\omega_0, z) \leqslant K_D(z_0, z) + K_D(z_0, \omega_0) \\ &\leqslant \frac{1}{2}\log\frac{1+r}{1-r} + C_2 - \frac{1}{2}\log \delta_D(z_0), \end{aligned}$$

for all $z \in B_D(z_0, r)$. Hence,

$$e^{2(C_1 - C_2)}\delta_D(z_0) \leqslant \frac{2}{1-r}\delta_D(z).$$

The left-hand inequality is obtained in the same way by reversing the roles of z_0 and z. □

Remark 2.4. Suppose $D = \{z : r(z) < 0\}$ is a bounded domain with smooth boundary, then $r(z) \approx \delta_D(z)$ uniformly in $z \in D$, thus after enlarging the constant C_3 if necessary, we also have for any $z_0 \in D$ and $z \in B_D(z_0, r)$,

$$\frac{C_3}{1-r}r(z_0) \geq r(z) \geq \frac{1-r}{C_3}r(z_0).$$

There is a precise description of the Kobayashi ball with respect to the minimal basis:

Theorem 2.5 Theorem 1, [15]. *If D is a bounded convex domain with smooth boundary of finite type, for any $r \in (0, 1)$ and $z \in D$,*

$$\frac{r}{n}\mathbb{D}^n(z, \sigma(z)) \subset B_D(z, r) \subset \frac{2r}{1-r}\mathbb{D}^n(z, \sigma(z)),$$

where $\mathbb{D}^n(z, \sigma(z)) = \{\omega : |\omega_i - z_i| < \sigma_i(z)\}$ with $\sigma_i(z)$ defined in (3).

We also note that the polydiscs have the following properties:

Lemma 2.6 Propositions 2.4, 2.5, [13]. *Suppose $z_0 \in W$. There is a constant $C_4 > 0$ such that if $z_0 \in V_j$ for any $j = 1, \ldots, K$, then*

$$\frac{1}{C_4}P^j(z_0, \epsilon) \subset P^j(z_0, 2\epsilon) \subset C_4 P^j(z_0, \epsilon).$$

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Moreover, if $z \in P^j(z_0, \epsilon)$, then there exists a constant $C_5 > 0$ such that

$$\frac{1}{C_5} P^j(z, \epsilon) \subset P^j(z_0, \epsilon) \subset C_5 P^j(z, \epsilon).$$

Lemma 2.7. *Let D be a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. Then, for every $r \in (0,1)$, there exist $M \in \mathbb{N}$ and a sequence of points $\{z_k\} \subset D$ such that $D = \bigcup_{k=0}^{\infty} B_D(z_k, r)$ and no point of D belongs to more than M of the balls $B_D(z_k, R)$, where $R = (1/2)(1+r)$.*

Proof. Let $\{B_j = B_D(z_j, \frac{r}{3})\}_{j\in\mathbb{N}}$ be a sequence of Kobayashi balls covering D. We can extract a subsequence $\{\Delta_k = B_D(z_k, r/3)\}_{k\in\mathbb{N}}$ of disjoint balls in the following way: Set $\Delta_1 = B_1$. Suppose that we have already chosen $\Delta_1, \dots, \Delta_l$. We define Δ_{l+1} as the first ball in the sequence $\{B_j\}$, which is disjoint from $\Delta_1 \cup \dots \cup \Delta_l$. In particular, by construction, every B_j must intersect at least one Δ_k.

We now claim that $\{B_D(z_k, r)\}_{k\in\mathbb{N}}$ is a covering of D. Indeed, let $z \in D$. Since $\{B_j\}_{j\in\mathbb{N}}$ is a covering of D, there is a $j_0 \in \mathbb{N}$ such that $z \in B_{j_0}$. As remarked above, we get $k_0 \in \mathbb{N}$ such that $B_{j_0} \cap \Delta_{k_0} \neq \emptyset$. By taking $w \in B_{j_0} \cap \Delta_{k_0}$, then we have

$$\rho_D(z, z_{k_0}) \leqslant \rho_D(z, w) + \rho_D(w, z_{k_0}) \leqslant \frac{2}{3} r,$$

and $z \in B_D(z_{k_0}, r)$.

To conclude the proof, we have to show that there is $m = m_r \in \mathbb{N}$ so that each point $z \in D$ belongs to at most m of the balls $B_D(z_k, R)$. Put $r_1 = \frac{1}{3}\min\{r, 1-r\}$ and $R_1 = \frac{1}{6}(5+r)$. Since $z \in B_D(z_k, R)$ is equivalent to $z_k \in B_D(z, R)$, we obtain that $z \in B_D(z_k, R)$ implies $B_D(z_k, r_1) \subset B_D(z, R_1)$. Therefore, noting that the balls $\{B_D(z_k, r_1)\}$ are pairwise disjoint, we deduce that

$$\operatorname{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} \leqslant \frac{\nu(B_D(z, R_1))}{\nu(B_D(z_k, r_1))}. \tag{5}$$

Case (1). Assume that $z \in W$. Then, from Theorem 2.5, it follows that

$$\begin{aligned}
\operatorname{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} &\leqslant \left(\frac{2nR_1}{r_1(1-R_1)}\right)^{2n} \frac{\nu(\mathbb{D}^n(z, \sigma(z)))}{\nu(\mathbb{D}^n(z_k, \sigma(z_k)))} \\
&= \left(\frac{2nR_1}{r_1(1-R_1)}\right)^{2n} \frac{\nu(P(z, |r(z)|))}{\nu(P(z_k, |r(z_k)|))} \\
&= \left(\frac{2nR_1}{r_1(1-R_1)}\right)^{2n} \frac{\nu(P(z, |r(z)|))}{\nu(P(z, |r(z_k)|))} \frac{\nu(P(z, |r(z_k)|))}{\nu(P(z_k, |r(z_k)|))}.
\end{aligned}$$

Noting that $z_k \in B_D(z, R)$, by Remark 2.4 and Lemma 2.6, it follows that

$$\operatorname{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} \leqslant \left(\frac{2nR_1}{r_1(1-R_1)}\right)^{2n} C_5 C_4^{|\log \frac{1-R}{C_3}|}.$$

Case (2). Assume that $z \in D \backslash W$. Since $D \backslash W$ is compact, there exists some $M > 0$ such that

$$\operatorname{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} \leqslant \frac{\nu(B_D(z, R_1))}{\nu(B_D(z_k, r_1))} \leqslant M.$$

□

Lemma 2.8. *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type. Then, for any $r \in (0,1)$ and $z_0 \in W$, we have*

$$\varphi(z_0) \leqslant \left(\frac{2n}{1-r}\right)^{2n} \cdot \frac{1}{\nu(B_D(z_0, r))} \int_{B_D(z_0, r)} \varphi \, d\nu$$

for all nonnegative plurisubharmonic functions $\varphi : D \to \mathbb{R}^+$.

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Proof. By applying the submean value property to each value separately, we deduce that on the polydisk $\mathbb{D}^n(z_0,\epsilon)=\{z:|z_i-z_{0,i}|<\epsilon_i\}\subset D$, it holds

$$\varphi(z_0)\leqslant\frac{1}{\nu(\mathbb{D}^n(z_0,\epsilon))}\int_{\mathbb{D}^n(z_0,\epsilon)}\varphi d\nu,$$

as φ is a plurisubharmonic function.

Theorem 2.5 implies that

$$\frac{r}{n}\mathbb{D}^n(z_0,\sigma(z_0))\subset B_D(z_0,r)\subset\frac{2r}{1-r}\mathbb{D}^n(z_0,\sigma(z_0)).$$

Thus,

$$\begin{aligned}\frac{1}{\nu(B_D(z_0,r))}\int_{B_D(z_0,r)}\varphi d\nu&\geq\left(\frac{1-r}{2r}\right)^{2n}\frac{1}{\nu(\mathbb{D}^n(z_0,\sigma(z_0)))}\int_{\frac{r}{n}\mathbb{D}^n(z_0,\sigma(z_0))}\varphi d\nu\\&\geq\left(\frac{1-r}{2n}\right)^{2n}\varphi(z_0),\end{aligned}$$

which completes the proof. □

Corollary 2.9. *Let $D\subset\subset\mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type. For any $r\in(0,1)$, denote $R=(1/2)(1+r)\in(0,1)$. Then, for any $z_0\in W$ and $z\in B_D(z_0,r)$, it holds that*

$$\varphi(z)\leqslant\left(\frac{8n^2r}{(1-r)^3}\right)^{2n}\frac{1}{\nu(B_D(z_0,r))}\int_{B_D(z_0,R)}\varphi d\nu$$

for every nonnegative plurisubharmonic function $\varphi:D\to\mathbb{R}^+$.

Proof. Let $r_1=\frac{1}{2}(1-r)$. By using the triangle inequality, $z\in B_D(z_0,r)$ yields that $B_D(z,r_1)\subset B_D(z_0,R)$. Lemma 2.8 and Theorem 2.5 then imply that

$$\begin{aligned}\varphi(z)&\leqslant\left(\frac{2n}{1-r}\right)^{2n}\frac{1}{\nu(B_D(z,r_1))}\int_{B_D(z,r_1)}\varphi d\nu\\&\leqslant\left(\frac{2n}{1-r}\right)^{2n}\frac{1}{\nu(B_D(z,r_1))}\int_{B_D(z_0,R)}\varphi d\nu\\&=\left(\frac{2n}{1-r}\right)^{2n}\frac{\nu(B_D(z_0,r))}{\nu(B_D(z,r_1))}\frac{1}{\nu(B_D(z_0,r))}\int_{B_D(z_0,R)}\varphi d\nu\\&\leqslant\left(\frac{8n^2r}{(1-r)^3}\right)^{2n}\frac{1}{\nu(B_D(z_0,r))}\int_{B_D(z_0,R)}\varphi d\nu,\end{aligned}$$

for all $z\in B_D(z_0,\ r)$. The proof is complete. □

2.4 | The Bergman kernel

Let $K:D\times D\to\mathbb{C}$ be the Bergman kernel of the domain D. It has the reproducing property; that is, for any $z\in D$,

$$f(z)=\int_D K(z,\zeta)f(\zeta)d\nu,\qquad\forall f\in A^2(D).$$

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Since $K(\cdot,\zeta)=\overline{K(\zeta,\cdot)}\in A^2(D)$, particularly we have

$$K(z,\,z)=\int_D |K(z,\,\zeta)|^2 d\nu(\zeta)=\|K(z,\,\cdot)\|_2^2.$$

For each $z_0\in D$, let $k_{z_0}\in A^2(D)$ denote the normalized Bergman kernel given by

$$k_{z_0}(z)=\frac{K(z,z_0)}{\|K(\cdot,z_0)\|_2}=\frac{K(z,z_0)}{\sqrt{K(z_0,z_0)}}.$$

Clearly, $\left\|k_{z_0}\right\|_2=1$. The Berezin transform $B\mu$ of a finite measure μ on the domain D is the function given by

$$B\mu(z)=\int_D \left|k_z(\zeta)\right|^2 d\mu(\zeta)$$

for all $z\in D$

Lemma 2.10 (Theorem 3.4, [13]). *Let $D\subset\subset\mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type, and let $V_j\subset U_j$ be the neighborhood of $p_j\in\partial D$ in Section 2.2. Then, if $z\in V_j\cap D$,*

$$K_D(z,z)\gtrsim\prod_{i=1}^n \tau_i^j(z,|r(z)|)^{-2}. \tag{6}$$

Note that we can take a constant C_6 uniformly in j, such that for any $z\in W\subset\bigcap_{j=1}^K V_j$,

$$K_D(z,z)\geq C_6\prod_{i=1}^n \sigma_i(z)^{-2}.$$

Lemma 2.11 (Theorem 5.2, [13]). *Let $D\subset\subset\mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type. Suppose that $V_j\subset U_j$ is the neighborhood of $p_j\in\partial D$. Then, for all multi-indices $\mu,\ \nu$, there exists a constant $C_{j,\mu,\nu}$ such that, for all $z,\ \omega\in V_j\cap D$,*

$$|D^\mu\bar{D}^\nu K_D(z,\omega)|\leq C_{j,\mu,\nu}\prod_{i=1}^n \tau_i^j(z,|r(z)|+|r(\omega)+M^j(z,\omega)|)^{-2-\mu_i-\nu_i},$$

where $M^j(z,\omega)=\inf\{\epsilon>0:\omega\in P_\epsilon^j(z)\}$.

First suppose $\omega\in B_D(z,r)$ for some $r<\frac{1}{3}$. Shrinking V_j as necessary, we may assume that $\omega\in V_j$. Then, by Theorem 2.5, it follows that

$$B_D(z,r)\subset\frac{2r}{1-r}\mathbb{D}^n(z,\sigma(z))\subset P^j_{|r(z)|}(z).$$

Therefore,

$$M^j(z,\omega)\leq|r(z)|.$$

By using Remark 2.4, we deduce that

$$|r(z)|\leq|r(z)|+|r(\omega)|+M^j(z,\omega)\leq\left(2+\frac{2}{3}C_3\right)|r(z)|.$$

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Then, Lemma 2.6 implies that

$$|D^{\mu}\bar{D}^{\nu}K_D(z,\omega)| \le C_{j,\mu,\nu}\prod_{i=1}^{n}\tau_i^j(z,|r(z)|+|r(\omega)+M^j(z,\omega)|)^{-2-\mu_i-\nu_i}$$

$$\le C_4^{\log_2(2+\frac{2}{3}C_3)}C_{j,\mu,\nu}\prod_{i=1}^{n}\tau_i^j(z,|r(z)|)^{-2-\mu_i-\nu_i}$$

$$\le C_{\mu,\nu}\prod_{i=1}^{n}\sigma_i(z)^{-2-\mu_i-\nu_i}, \tag{7}$$

where $\omega \in B_D(z,r)$ and $C_{\mu,\nu} = \max\left\{C_4^{\log_2(2+\frac{2}{3}C_3)}C_{j,\mu,\nu} \ : \ j = 1\cdots K\right\}$.

Lemma 2.12. *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type. There exists r_0 such that, $\forall z_0 \in W$, $r \in (0,r_0)$, and $\omega \in B_D(z_0, r)$,*

$$K_D(z,\omega) \ge \frac{C_6}{2}\prod_{i=1}^{n}\sigma_i(z)^{-2}. \tag{8}$$

Proof. By Theorem 2.5, we have

$$\frac{|(\omega_i - z_i)|}{\sigma_i(z)} \le \frac{2r}{1-r}, \quad i = 1\cdots n. \tag{9}$$

Denote $K_D(z,\cdot) = f(\cdot) + ig(\cdot)$, where f and g are real functions on convex domains D. By applying the Lagrange's mean value theorem to f and g, respectively, there exist θ_1, $\theta_2 \in [0,1]$ such that if we write $z_k = x_k + ix_{n+k}$ and $\omega_k = \xi_k + i\xi_{n+k}$, then

$$f(\omega) - f(z) = \sum_{k=1}^{2n}\frac{\partial f}{\partial x_k}(z+\theta_1(\omega - z))(\xi_k - x_k)$$

and

$$g(\omega) - g(z) = \sum_{k=1}^{2n}\frac{\partial g}{\partial x_k}(z+\theta_2(\omega - z))(\xi_k - x_k).$$

Noting that, for f (or g), we have

$$\left|\frac{\partial f}{\partial x_k}\right| + \left|\frac{\partial f}{\partial x_{n+k}}\right| \le 4\left|\frac{\partial f}{\partial \bar{z}_k}\right|, \quad \left|\frac{\partial g}{\partial x_k}\right| + \left|\frac{\partial g}{\partial x_{n+k}}\right| \le 4\left|\frac{\partial g}{\partial \bar{z}_k}\right|.$$

By denoting $C = \max\{C_{\mu,\nu} \ : \ |\mu| = 0, |\nu| = 1\}$ and using (7) and (9), we have

$$|K_D(z,\omega) - K_D(z,z)| \le |f(\omega) - f(z)| + |g(\omega) - g(z)|$$

$$\le 8C\sum_{k=1}^{n}\prod_{i=1}^{n}\sigma_i(z)^{-2}\sigma_k(z)^{-1}\frac{2r}{1-r}\sigma_k(z)$$

$$\le C\frac{16nr}{1-r}\prod_{i=1}^{n}\sigma_i(z)^{-2}.$$

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Therefore, if we choose r_0 small enough such that

$$C\frac{16nr_0}{1-r_0} < \frac{C_6}{2},$$

then for any $r \in (0,\ r_0)$ and $\omega \in B_D(z,r)$, we have the desired estimate

$$\begin{aligned} K_D(z,\omega) &\geq K_D(z,z) - |K_D(z,z) - K_D(z,\omega)| \\ &\geq \frac{C_6}{2}\prod_{i=1}^{n}\sigma_i(z)^{-2}. \end{aligned}$$

□

By Lemma 2.10 and Lemma 2.12, the following corollary is direct:

Corollary 2.13. *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain with smooth boundary of finite type. There exist $r_0 \in (0,1)$ and $C_7 > 0$ such that if $z_0 \in W$, $\forall r \in (0,\ r_0)$, $z \in B_D(z_0,r)$, then*

$$\left|k_{z_0}(z)\right|^2 \geqslant C_7\prod_{i=1}^{n}\sigma_i(z_0)^{-2}.$$

Proof. By taking $\mu = \nu = 0$ in Lemma 2.11, the inequality follows from the above Lemma 2.12. □

3 | CARLESON MEASURE

Let $D \subset\subset \mathbb{C}^n$ be a convex domain with smooth boundary of finite type. The Bergman space $A^2(D)$ of D is the Banach space of holomorphic L^2-functions on D, that is, $A^2(D) = L^2(D) \cap \mathcal{O}(D)$, endowed with the L^2-norm

$$\|f\|_2^2 = \int_D |f(z)|^2 d\nu,$$

where ν is the Lebesgue measure normalized with $\nu(B_1(0)) = 1$. A finite positive Borel measure μ on D is said to be a Carleson measure of $A^2(D)$ if there exists $C > 0$ such that

$$\int_D |f(z)|^2 d\mu \leqslant C\|f\|_2^2, \quad \forall f \in A^2(D).$$

Theorem 3.1. *Let μ be a finite positive Borel measure on a bounded convex domain $D \subset \mathbb{C}^n$ with smooth boundary of finite type. Then, the following statements are equivalent:*

(1) μ is a Carleson measure of $A^2(D)$;
(2) the Berezin transform of μ is bounded;
(3) for some (and hence for any) $r \in (0, r_0)$, there exists $C_r > 0$ such that, for all $z \in D$,

$$\mu(B_D(z,r)) \leqslant C_r\nu(B_D(z,r)).$$

The proof of Theorem 3.1 follows from Corollary 2.9, Theorem 2.5, and Corollary 2.13.

Proof. (1)⇒(2):

$$B\mu(z_0) = \int_D \left|k_{z_0}(z)\right|^2 d\mu(z) \leqslant C\left\|k_{z_0}\right\|_2^2 = C.$$

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(2)⇒(3): If $z_0 \notin W$, where W is defined in (1) in the Preliminaries 2.2, then we have

$$\mu(B_D(z_0,r)) \le \mu(D) \le \left(\frac{n}{r}\right)^{2n} \frac{\mu(D)}{\prod_{i=1}^{n} \sigma_i^2(z_0)} \nu(B_D(z_0,r))$$

$$\le \left(\frac{n}{r}\right)^{2n} \frac{\mu(D)}{\delta_D(z_0)^{2n}} \nu(B_D(z_0,r)).$$

If $z_0 \in V_j$ for some $j \in \{1,2,\dots,K\}$, noting that the Berezin transform is bounded, then there exists $C > 0$ independent of z_0 and r such that

$$\int_{B_D(z_0,r)} \left|k_{z_0}(z)\right|^2 d\mu(z) \leqslant B\mu(z_0) \leqslant C.$$

Then, by Corollary 2.13,

$$C_7 \prod_{i=1}^{n} \sigma_i(z_0)^{-2} \mu(B_D(z_0,r)) \le \int_{B_D(z_0,r)} \left|k_{z_0}(z)\right|^2 d\mu(z) \leqslant C.$$

Thus, by Theorem 2.5, we obtain

$$\mu(B_D(z_0,r)) \le \frac{C}{C_7} \prod_{i=1}^{n} \sigma_i(z_0)^2 \le \left(\frac{n}{r}\right)^{2n} \frac{C}{C_7} \nu(B_D(z_0,r)).$$

(3)⇒(1): Let $\{z_k\}$ be the sequence given by Lemma 2.7. Clearly, for all $f \in A^2(D)$, we have

$$\int_D |f(z)|^2 d\mu(z) \leqslant \sum_{k=1}^{\infty} \int_{B_D(z_k,r)} |f(z)|^2 d\mu(z).$$

Since $|f|^2$ is a nonnegative plurisubharmonic function, let $K_r = \left(\frac{8n^2 r}{(1-r)^3}\right)^{2n}$ and by Lemma 2.9,

$$\int_{B_D(z_k,r)} |f(z)|^2 d\mu(z) \leqslant \frac{K_r}{\nu(B_D(z_k,r))} \int_{B_D(z_k,r)} d\mu(z) \int_{B_D(z_k,R)} |f(\zeta)|^2 d\nu(\zeta)$$

$$= K_r \frac{\mu(B_D(z_k,r))}{\nu(B_D(z_k,r))} \int_{B_D(z_k,R)} |f(\zeta)|^2 d\nu(\zeta)$$

$$\leqslant K_r C_r \int_{B_D(z_k,R)} |f(\zeta)|^2 d\nu(\zeta),$$

where $R = (1/2)(1+r)$. Hence,

$$\int_D |f(z)|^2 d\mu(z) \leqslant K_r C_r \sum_{k=1}^{\infty} \int_{B_D(z_k,R)} |f(\zeta)|^2 d\nu(\zeta) \leqslant K_r C_r M \|f\|_2^2,$$

as desired. □

4 | UNIFORMLY SEQUENCES

Definition 4.1 (Uniformly discrete sequence). Suppose that (X, d) is a metric space. A sequence $\Gamma = \{x_i\} \subset X$ of points is uniformly discrete if there exists $\delta > 0$ such that $d(x_i, x_j) \geqslant \delta$ for all $i \neq j$. And, we call $\inf_{i \neq j} d(x_i, x_j)$ the separation constant of Γ.

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Moreover, given $x_0 \in X$, $r > 0$, and a subset $\Gamma \subset X$, we denote by $M(x_0, r, \Gamma)$ the number of points of Γ contained in the ball centered at x_0 with radius r.

The following result is due to M. Abate and A. Saracco. And, we add their proof here for the sake of completeness.

Lemma 4.2 [4], Lemma 4.1. *Suppose that X is a metric space and $\Gamma = \{x_n\}_{n\in\mathbf{N}} \subset X$ is a sequence in X. If there exists $M \geqslant 1$ and $r > 0$ such that $M(x, r, \Gamma) \leqslant M$ for all $x \in X$, then Γ is the union of at most M uniformly discrete sequences.*

Proof. We will show that $\Gamma = \Gamma_0 \cup \ldots \cup \Gamma_{M-1}$. where Γ_i are uniformly discrete sequences.

First of all, suppose that $\Gamma_0 = \cdots = \Gamma_{M-1} = \emptyset$. By induction on $n \geq 0$, we put each x_n in a specified Γ_j. Denote by $B(x, r)$ the metric ball with center x and radius r. Then, we write $m(x_i) = j$ if $x_i \in \Gamma_j$.

Furthermore, we can set $x_0 \in \Gamma_0$. If we have already defined $m(x_i)$ for $i \leqslant n$, then we consider x_{n+1}. Therefore, $\Gamma \cap B(x_{n+1}, r)$ contains no more than M points, and one of them is x_{n+1}. Thus, we deduce that $\{x_0, \ldots, x_n\} \cap B(x_{n+1}, r)$ contains no more than $M - 1$ points.

Then, we can write

$$m(x_{n+1}) = \min\{i \in \{0, \ldots, M-1\} \mid i \neq m(x_j)\}$$

for all $0 \leqslant j \leqslant n$ such that $x_j \in B(x_{n+1}, r)\}$. As a result, we obtain that $d(x_{n+1}, x_j) \geqslant r$ for all $x_j \in \Gamma_{m(x_{n+1})}$ with $j < n + 1$. By construction, if x_h, $x_k \in \Gamma_j$ with $h > k$, then we obtain $d(x_h, x_k) \geqslant r$.

In conclusion, $\Gamma_0, \ldots, \Gamma_{M-1}$ are uniformly discrete sequences and we complete the proof. □

Theorem 4.3. *Let $D \subset\subset \mathbb{C}^n$ be a convex bounded domain with boundary of finite type, and let $\Gamma = \{z_k\}$ be a sequence in D. Then, the following statements are equivalent:*

(1) Γ is a finite union of uniformly discrete (with respect to the Kobayashi distance) sequences;
(2) $\sum_{z_k\in\Gamma} \prod_{i=1}^n \sigma_i^2(z_k)\delta_{z_k}$ is a Carleson measure of $A^2(D)$;
(3) $\sum_{z_k\in\Gamma} \prod_{i=1}^n \sigma_i^2(z_k)|f(z_k)|^2 \leq C\|f\|_2^2$,

where δ_{z_k} is the Dirac measure at z_k.

Proof. (1) ⇒ (3) : It suffices to prove the assertion when Γ is a single uniformly discrete sequence. Let $\delta > 0$ be the separation constant of Γ, and put $r = \delta/2 \wedge r_0$. By the triangle inequality, the Kobayashi balls $B_D(z_k, r)$ are pairwise disjoint. Hence,

$$\int_D |f(z)|^2 d\nu \geqslant \sum_{z_k\in\Gamma} \int_{B_D(z_k,r)} |f(z)|^2 d\nu.$$

Note that $|f|^2$ is plurisubharmonic and nonnegative; from Lemma 2.8, it follows that

$$|f(z_k)|^2 \leqslant \left(\frac{2n}{1-r}\right)^{2n} \frac{1}{\nu(B_D(z_k, r))} \int_{B_D(z_k,r)} |f|^2 d\nu.$$

Then by Theorem 2.5, we have

$$\sum_{z_k\in\Gamma} \prod_{i=1}^n \sigma_i^2(z_k)|f(z_k)|^2 \leq \left(\frac{2n^2}{r(1-r)}\right)^{2n} \|f\|_2^2.$$

(3) ⇒ (2): The proof is obvious.
(2) ⇒ (1): By Corollary 2.13, for any $z_0 \in W$,

$$\left|k_{z_0}(z)\right|^2 \geqslant C_7 \prod_{i=1}^n \sigma_i(z_0)^{-2}.$$

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Hence,

$$M(z_0, r, \Gamma) \le \frac{1}{C_7} \sum_{z \in B_D(z_0, r) \cap \Gamma} \left| k_{z_0}(z) \right|^2 \prod_{i=1}^{n} \sigma_i(z_0)^2$$
$$\le \frac{C}{C_7} \| k_{z_0}(z) \|_2^2 = \frac{C}{C_7}$$

for some $C > 0$.

For $z \in D/W$, it is easy to see that $M(z_0, r, \Gamma) < \infty$. Together with Lemma 4.2, we thus complete the proof. □

ACKNOWLEDGMENTS

The first author is supported by NSFC (Numbers 11901205 and 12071155). The second author is supported by National Key R&D Program of China (Number 2021YFA1003100), NSFC (Number 11925107), and Key Research Program of Frontier Sciences, CAS (Number ZDBS-LY-7002). The third author is supported by the Young Scientist Program of the Ministry of Science and Technology of China (Number 2021YFA1002200).

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How to cite this article: H. Li, J. Liu, and H. Wang, *Carleson measures on convex domains with smooth boundary of finite type*, Math. Nachr. **297** (2024), 694–706. https://doi.org/10.1002/mana.202200325

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J. Nonlinear Var. Anal. 9 (2025), No. 2, pp. 247-262
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.9.2025.2.05

THE THEORY OF HARDY-SOBOLEV SPACES OVER THE UPPER HALF-PLANE

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Abstract. We mainly investigate some results of Hardy-Sobolev spaces on the upper half-plane. In this paper, the Paley-Wiener theorem of Hardy-Sobolev spaces is established. The theorem asserts that the reproducing kernel of the Hardy-Sobolev spaces can be found. On account of this, the estimation of the reproducing kernel is produced. We also consider the multiplication operators on the spaces, and the spectrum of multiplication operators is characterized. In addition, the condition for the boundedness of weighted composition operators can be founded by applying the property of self-maps and the reproducing kernel.

Keywords. Holomorphic Fourier transform; Hardy-Sobolev spaces; Paley-Wiener theorem; Reproducing kernel; Weighted composition operator.

1. INTRODUCTION

Function spaces, as a branch of functional analysis, has been widely developed. Classical Hardy spaces have formed a rich theoretical system [1, 20]. It consists of functions in the Lebesgue spaces L^p that are smooth inside the domain and have finite integral growth on the boundary. Sobolev developed a novel kind of function spaces during his investigation of the elastic wave problem in the 1930s. This kind of spaces later came to be known as Sobolev spaces, which are Banach space constructed by weakly differentiable functions. In harmonic analysis and partial differential equations theory, Hardy spaces are the best alternative to Lebesgue spaces $L^p(0 < p \leq 1)$ and the Sobolev space requires that the weak derivative of the function belongs to $L^p(p > 1)$ [2]. As a result, it makes sense to further research Hardy-Sobolev spaces, which demand that a function's derivative also belongs to Hardy spaces. Hardy-Sobolev spaces include a number of well-known classical function spaces, such as Hardy spaces, Bergmann spaces, Dirichlet spaces, and so on. Since Hardy-Sobolev spaces have more sophisticated spatial structure, they have more deep outcomes than generic classical spaces.

In recent years, scholars studied the Hardy-Sobolev spaces defined on the unit disk or unit ball. He-Cao-Zhu [3] obtained some growth estimates for Hardy-Sobolev functions of the unit

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Received 24 March 2024; Accepted 23 April 2024; Published online 1 January 2025.

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ball and gave the representation of the spectra of composition operators and described the Fredholmness of composition operators equivalently. Ortega-Fàbrega [4] gave characterizations of pointwise multipliers of the holomorphic Hardy-Sobolev spaces on the unit ball of $\mathbb{C}^n$. In [5–7], the multipler algebra on Hardy-Sobolev spaces over unit disks and unit balls was studied with computing the spectrum and essential spectrum of multiplication operators. There are fewer tools available on the half-planes, However. The theory of Hardy-Sobolev spaces over the half-plane or tubes still remains to be studied. Espallargas [8] and Galé et al. [9] established the extended form of the Paley-Wiener theorem for Hardy-Sobolev spaces on the right-hand half-plane and derived a series of results. Cho-Kim [21] considered Hardy–Sobolev spaces for high dimension case and established an atomic decomposition theorem, analogous to the atomic decomposition characterization of Hardy spaces. Rotkevich [22,23] used the method of pseudoanalytic continuation to obtain the characterization of Hardy–Sobolev spaces on strongly convex domains in terms of polynomial approximations, and then promoted the related conclusions to strictly pseudoconvex domains. Observe that Hardy-Sobolev spaces have wide applications in the areas, such as PDE, wavelet analysis, signal analysis, and so on. For example, Ward-Partington [10] established rational wavelet decompositions of the Hardy–Sobolev classes on the half-plane by using Daubechies wavelet theory. Dang-Qian-Yang [11] extended some formulas to signals in Sobolev spaces by introducing Hardy-Sobolev derivatives.

In this paper, we focus on the function and operator theories of Hardy-Sobolev spaces. More precisely, this paper mainly considers the problems such as the Paley-Wiener theorem, reproducing kernels, multiplication operators, and weighted composition operators. These problems could supplement each other and are important for the development of Hardy-Sobolev spaces. It is well known that the classical Paley–Wiener theorem has important applications in the function theory of complex variables and approximation of real variables. Due to the property of isometric isomorphism, Paley-Wiener type theorem provides effective tools for studying these two related function spaces. Functions with compact and convex spectrum on $\mathbb{R}^n$ were described by the classical Paley-Wiener theorem through the properties of their analytic extensions in the complex domain; see [24]. In addition, the good reproducibility of the reproducing kernel has brought great convenience to many fitting and reconstruction problems in mathematics and engineering, and has also given rise to the great development of many new interdisciplinary fields. Elements in the reproducing kernel spaces can be generated by the inner product with the reproducing kernel, which determines projections, which are the basic method for discussing optimal approximation problems in inner product spaces, which determines the great convenience of reproducing kernels in various numerical analysis, data processing, and optimization problems.

Weighted composition operators, which are a generalized composition operators, are the combination of composition operators and multiplication operators. Investigating weighted composition operators is of great significance for studying harmonic analysis, functional analysis, operator algebra, and topological dynamical systems. It is important to develop the Paley-Wiener type theorem, reproducing kernel and operator theories in Hardy-Sobolev spaces. Our main results are as follows.

A. Let $\mathscr{F}$ be the holomorphic Fourier transform $\mathscr{F} : L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$. There exists $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$, and then $H_2^{(n)}$ is a Hilbert space endowed with inner product

$$\langle F, G\rangle_{H_2^{(n)}} = \int_{-\infty}^{+\infty} x^{2n}(F^*)^{(n)}(x)\overline{(G^*)^{(n)}(x)}dx, \qquad F, G \in H_2^{(n)},$$

and its associated norm $\|F\|_{H_2^{(n)}} = \|z^n F^{(n)}\|_{H_2}$, $F \in H_2^{(n)}$. Moreover, $\mathscr{F}$ is an isometric isomorphism from $\mathscr{T}_2^{(n)}$ onto $H_2^{(n)}$, i.e., $F, G \in H_2^{(n)}$ if and only if there exists $f, g \in \mathscr{T}_2^{(n)}$ and $\langle f, g\rangle_{\mathscr{T}_2^{(n)}} = \langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{H_2^{(n)}}$.

B. Let n be a positive integer. Then the function K_n defined on $\mathbb{C}^+ \times \mathbb{C}^+$ by

$$K_n(z,w) = \frac{1}{((n-1)!)^2}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{i}{yz - x\overline{w}}dxdy, \qquad z,w \in \mathbb{C}^+,$$

is the reproducing kernel of $H_2^{(n)}$, i.e., if $K_{n,w}(z) = K_n(z,w)$, then $K_{n,w} \in H_2^{(n)}$ and $F(w) = \langle F, K_{n,w}\rangle_{H_2^{(n)}}$ for all $F \in H_2^{(n)}$.

C. Let n be the positive integer. Then, for $z \in \mathbb{C}^+$,

$$\frac{1}{(n-1)!\sqrt{2n-1}}\frac{1}{\sqrt{|z|}} \le \|K_{n,z}\|_{H_2^{(n)}} \le \frac{\sqrt{\pi}}{(n-1)!\sqrt{n}}\frac{1}{\sqrt{|z|}}.$$

D. For any $\varphi \in \mathscr{M}_n$, the spectrum of the multiplication operator $M_\varphi : H_2^{(n)} \to H_2^{(n)}$ is given by $\sigma(M_\varphi) = \overline{\varphi(\mathbb{C}^+)}$, which is the closure of the range of φ in the complex plane.

E. If $\psi \in \mathscr{M}_n$ satisfying that there exists a positive consant δ such that $|\psi(z)| \ge \delta$ for all $z \in \mathbb{C}^+$ and $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ for $n \ge 1$, then the self-map φ satisfies $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.

F. Let $\psi \in H_\infty^{(n)}$ and φ be an analytic self-map on $\mathbb{C}^+$ satisfying that $\varphi'(\infty) < \infty$ and $\sup_{z\in\mathbb{C}^+}|z^k\frac{\varphi^{(k)}(z)}{\varphi(z)}| < \infty$, $k = 1,2,\ldots,n$. Then the weighted composition operator $C_{\psi,\varphi} : H_2^{(n)} \to H_2^{(n)}$ is bounded.

The present paper is organized as follows. In Section 2, we introduce some notations and definitions, which are necessary in this paper. In Section 3, we establish the Paley-Wiener type theorem of Hardy-Sobolev spaces. In Section 4, we compute the reproducing kernel on the spaces and give the estimation of the reproducing kernel additionally. In Section 5, the multiplication operators and weighted composition operators on the spaces is discussed. We give some properties about the multiplication operators and establish the spectrum of the operator. In addition, we consider the condition for the boundedness of weighted composition operators, which are related to the reproducing kernel, multiplication operators and self-map.

2. PRELIMINARIES

Let $\mathbb{R}^+$ be the interval (0,∞ 0) and $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ denote the upper half-plane. Let $L^p(\mathbb{R}^+)$ be the usual Lebesgue space on (0,∞) with its norm

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty, \qquad f \in L^p(\mathbb{R}^+).$$

In particular, $L^2(\mathbb{R}^+)$ is a Hilbert space with its inner product given by $\langle f, g\rangle_2 = \int_0^\infty f(t)\overline{g(t)}dt$ for all $f, g \in L^2(\mathbb{R}^+)$. One usually writes $\langle\cdot,\cdot\rangle$ to denote the inner product of Hilbert spaces. Recall that the classical Hardy Space $H_2(\mathbb{C}^+)$ is the space of holomorphic functions on the

upper half-plane $\mathbb{C}^+$ such that

$$\|f\|_{H_2} = \sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx < \infty.$$

In fact, $H_2(\mathbb{C}^+)$ is a Hilbert space with inner product $\langle f, g\rangle_{H_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f^*(x)\overline{g^*(x)}dx$ for all $f, g \in H_2(\mathbb{C}^+)$, where $f^*(x) = \lim_{y\to 0} f(x+iy)$ a.e. $x \in \mathbb{R}$. In the case of unit disks, Hardy space $H_2(\mathbb{D})$ is endowed with the norm

$$\|F\|_{H_2(\mathbb{D})} = \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}}, \qquad F \in H_2(\mathbb{D}).$$

$\mathscr{T}_p^{(n)} (1 \le p < \infty)$ consist of all the measurable functions on $\mathbb{R}^+$ which belongs to the closure of $C_c^\infty(\mathbb{R}^+)$ corresponding to the norm

$$\|f\|_{\mathscr{T}_p^{(n)}} = \left(\int_0^\infty |f^{(n)}(t)t^n|^p dt\right)^{\frac{1}{p}} < \infty, \qquad f \in C_c^\infty(\mathbb{R}^+).$$

In particular, $\mathscr{T}_1^{(n)}$, a subalgebra of $L^1(\mathbb{R}^+)$, is a convolution Banach algebra with norm $\|\cdot\|_{\mathscr{T}_1^{(n)}}$. It was first introduced to approach Cauchy problems in [12,13] and it was further studied in [14]. For $p = 2$, it was proved that the range $\mathscr{L}(\mathscr{T}_2^{(n)})$, where $\mathscr{L}$ denotes to the Laplace transform, is an isometric isomorphism onto the Hardy-Sobolev space on the right half-plane [8].

In this paper, we consider the holomorphic Fourier transform $\mathscr{F}$ acting on the space $\mathscr{T}_2^{(n)}$, where $\mathscr{F}$ is given by $\mathscr{F}(f)(z) = \int_0^\infty f(t)e^{itz}dt$, $f \in L^2(\mathbb{R}^+), z \in \mathbb{C}^+$. Let f be any function in $L^2(\mathbb{R}^+)$ and $\mathscr{F}(f)(z) = F(z) = \int_0^\infty f(t)e^{itz}dt$, $z \in \mathbb{C}^+$. If $z = x+iy \in \mathbb{C}^+$, then $|e^{itz}| = e^{-ty}$, which shows that the integral above exists as a Lebesgue integral. By applying a series of theorems about real and complex analysis, it is easy to see that F is a holomorphic function on $\mathbb{C}^+$. The classical Paley-Wiener theorem [17] shows that the holomorphic Fourier transform $\mathscr{F} : L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$ is an isometric isomorphism, i.e., $F \in H_2(\mathbb{C}^+)$ if and only if there exists an $f \in L^2(\mathbb{R}^+)$ such that $F = \mathscr{F}(f)$ and $\|F\|_{H_2} = \|f\|_2$. Sometimes, one writes H_2 for short to denote the Hardy space on the upper half-plane.

Let X be a set. Recall that $\mathscr{H}$ is a reproducing kernel Hilbert space, or, more briefly, a RKHS on X if $\mathscr{H}$ is a Hilbert space of all complex functions on X and satisfies the condition that the evaluation functional, $E_x : \mathscr{H} \to \mathbb{C}$, defined by $E_x(f) = f(x)$, is bounded for every $x \in X$. If $\mathscr{H}$ is an RKHS on X, then an application of the Riesz representation theorem shows that the linear evaluation functional is given by the inner product with a unique vector in $\mathscr{H}$. Thus, for each $x \in X$, there exists a unique vector $k_x \in \mathscr{H}$ such that, for every $f \in \mathscr{H}, f(x) = E_x(f) = \langle f, k_x\rangle$. The function k_x is called a kernel-function and the span generated by all of kernel-functions is dense in $\mathscr{H}$. The function $K : X \times X \to \mathbb{C}$ defined by $K(x,y) = k_y(x) = \langle k_y, k_x\rangle, x, y \in X$ is said to be the reproducing kernel of $\mathscr{H}$ [15]. In fact, H_2 is a RKHS with its reproducing kernel given by $K(z,w) = K_w(z) = \frac{i}{z-\overline{w}}$, $z, w \in \mathbb{C}^+$. In addition, a kernel function K is said to be nonnegative, positive, definite or sometimes positive semidefinite meaning that $\sum_{i,j=1}^n \alpha_i\overline{\alpha_j}K(x_i,x_j) \ge 0$, $n \ge 1$, $\alpha_i, \alpha_j \in \mathbb{C}$, $x_i, x_j \in X$, $i, j = 1, 2, \ldots, n$. Let $\mathscr{S}$ be the class of all analytic self-maps of $\mathbb{C}^+$. The definition of the angular derivative on the unit disc $\mathbb{D}$ and $\mathbb{C}^+$ was introduced in [18]. In this paper, we mainly consider the self-map φ on the upper half-plane which fixes the point at infinity and has a finite angular derivative at ∞ in the sense

of Carathéodory. It means that $\varphi(\infty)=\infty$ and $\varphi'(\infty)<\infty$, where $\varphi(\infty)=\lim_{|z|\to\infty}\varphi(z)$ and $\varphi'(\infty)=\lim_{z\to\infty}\frac{z}{\varphi(z)}$, and the second limit above represent the non-tangential limit.

Here we use some notations and terminologies that are appeared in [8, 9]. We use W^{-1} to denote the integration operator on the space $C_c^\infty(\mathbb{R}^+)$, which is given by

$$W^{-1}\varphi(t)=\int_t^\infty \varphi(s)ds, \qquad \varphi\in C_c^\infty(\mathbb{R}^+), \quad t\geq 0.$$

For any function $\varphi\in C_c^\infty(\mathbb{R}^+)$, it is easy to show that $W^{-1}\varphi\in C_c^\infty(\mathbb{R}^+)$ so that we can define the integration operator $W^{-n}=W^{-1}(W^{-(n-1)})$ for every positive integer n. By applying Fubini's theorem, we have

$$W^{-n}\varphi(t)=\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\varphi(s)ds, \qquad \varphi\in C_c^\infty(\mathbb{R}^+), \quad t\geq 0.$$

In fact, the index n can be extended to any positive real number α. For more details about $W^{-\alpha}$, we refer to [8, 25].

Next, we consider the operator W^{-n} acting on the Hilbert space $L^2(t^n)$. For $n\in\mathbb{N}$, let $L^2(t^n)$ be the space of all complex measurable functions on $\mathbb{R}^+$ whose norm satisfies

$$\|\varphi\|_{L^2(t^n)}=\left(\int_0^\infty |\varphi(t)t^n|^2dt\right)^{\frac{1}{2}}<\infty.$$

Taking $m\in\mathbb{N}$ and Borel function φ, we have $W^{-n}\varphi\in L^2(\mathbb{R}^+)$ for all $\varphi\in L^2(t^n)$ by applying Hardy's inequality [16]

$$\int_0^\infty \left(W^{-m}\varphi(t)\right)^2dt\leq\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}\right)^2\int_0^\infty (t^m\varphi(t))^2dt.$$

As a matter of fact, the integration operator $W^{-n}:L^2(t^n)\to L^2(\mathbb{R}^+)$ is injective, which enables us to define Hardy-Lebesgue space $\mathscr{T}_2^{(n)}$. For $n\in\mathbb{N}$, let Hardy-Lebesgue spaces $\mathscr{T}_2^{(n)}$ be the range of operator W^{-n} on $L^2(t^n)$, i.e., $\mathscr{T}_2^{(n)}=W^{-n}(L^2(t^n))$, which is a subspace of $L^2(\mathbb{R}^+)$. It is clear that $\mathscr{T}_2^{(n)}$ is a Hilbert space with inner product $\langle f,g\rangle_{\mathscr{T}_2^{(n)}}=\int_0^\infty f^{(n)}(t)\overline{g^{(n)}(t)}t^{2n}dt$, $f,g\in\mathscr{T}_2^{(n)}$. For $n\geq k\geq 0$, let $\varphi(t)=|f^{(n)}(t)|t^k$ and $m=n-k$. By using Hardy's inequality, we have that the size of the space $\mathscr{T}_2^{(n)}$ decreases as n increases, i.e., $\mathscr{T}_2^{(n)}\subseteq\mathscr{T}_2^{(k)}\subseteq L^2(\mathbb{R}^+)$, $n\geq k\geq 0$. For $n\in\mathbb{N}$, let Hardy-Sobolev spaces $H_2^{(n)}$ be the linear space of all holomorphic functions F on $\mathbb{C}^+$ such that $z^kF^{(k)}\in H_2$, $k=0,1,\ldots,n$. It is clear that $H_2^{(n)}\subseteq H_2^{(0)}=H_2$. Also, we write $H_\infty^{(n)}$ to denote the linear space of all holomorphic functions F on $\mathbb{C}^+$ satisfying $z^kF^{(k)}\in H^\infty, \qquad k=0,1,\ldots,n$, where H^∞ denotes the bounded analytic function space on the upper half-plane. Since $H_2^{(n)}$ can be embedded into H_2, it follows that the evaluation functional on $H_2^{(n)}$ is also bounded, so it is a RKHS as well [9]. The multiplier algebra of $H_2^{(n)}$ is the set of all functions φ analytic on $\mathbb{C}^+$ such that $\varphi f\in H_2^{(n)}$ whenever $f\in H_2^{(n)}$. We denote it by $\mathscr{M}_n$. Given $\varphi\in\mathscr{M}_n$, we write $M_\varphi:H_2^{(n)}\to H_2^{(n)}$ for the multiplication operator $M_\varphi(f)=\varphi f \qquad (f\in H_2^{(n)})$. The weighted composition operator with the symbol ψ,φ acting on some function spaces is given by $C_{\psi,\varphi}f=\psi f\circ\varphi$.

3. Paley-Wiener Theorem

Paley-Wiener type theorems play an important role in the classical function theory. Now, we obtain the Paley-Wiener theorem of Hardy-Sobolev spaces as follows.

Theorem 3.1. *(Paley-Wiener) Let $\mathscr{F}$ be the holomorphic Fourier transform $\mathscr{F}: L^2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$. Then there exists $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$, and $H_2^{(n)}$ is a Hilbert space endowed with inner product $\langle F, G\rangle_{H_2^{(n)}} = \int_{-\infty}^{+\infty} x^{2n}(F^*)^{(n)}(x)\overline{(G^*)^{(n)}(x)}dx$, $F, G \in H_2^{(n)}$, corresponding to the norm $\|F\|_{H_2^{(n)}} = \|z^n F^{(n)}\|_{H_2}$, $F \in H_2^{(n)}$. Moreover, $\mathscr{F}$ is an isometric isomorphism from $\mathscr{T}_2^{(n)}$ onto $H_2^{(n)}$, i.e., $F, G \in H_2^{(n)}$ if and only if there exists $f, g \in \mathscr{T}_2^{(n)}$ and $\langle f, g\rangle_{\mathscr{T}_2^{(n)}} = \langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{H_2^{(n)}}$.*

In order to prove Theorem 3.1, we need the help of following lemmas.

Lemma 3.1. *[9] For $n \in \mathbb{N}$ and $\varphi \in L^2(t^n)$, we have, for $1 \le k \le n$,*

$$W^{-k}\varphi(t) = \frac{1}{(k-1)!}\int_t^\infty (s-t)^{n-1}\varphi(s)ds, \qquad t > 0.$$

Moreover, $W^{-k}\varphi$ is $(k-1)$-times differentiable with $(W^{-k}\varphi)^{(l)} = (-1)^l W^{-(k-l)}\varphi$ for every $1 \le l \le k-1$, and $(-1)^k(W^{-k}\varphi)^{(k)} = \varphi$.

Using Holder's inequality and taking $\varphi = (-1)^n f^{(n)}$ in Lemma 3.1, we obtain

$$|f^{(k)}(t)| \le C_{n,k} t^{-k-\frac{1}{2}}\|f\|_{\mathscr{T}_2^{(n)}}, \qquad f \in \mathscr{T}_2^{(n)}, \quad 0 \le k \le n-1, \quad t > 0,$$

where $C_{n,k}$ is a constant related to n and k.

The next lemma characterizes the equivalent inner-product of spaces $\mathscr{T}_2^{(n)}$.

Lemma 3.2. *[9] Let $f, g \in \mathscr{T}_2^{(n)}$. Then $\langle f, g\rangle_{\mathscr{T}_2^{(n)}} = \langle (t^n f)^{(n)}, (t^n g)^{(n)}\rangle_2$, where $\langle f, g\rangle_2 = \int_0^\infty f(t)\overline{g(t)}dt$.*

Lemma 3.3. *[9] For $n \ge 0$, the $(n+1)$-square matrix $C_n = (c_{i,j})_{0\le i,j\le n}$ defined by*

$$c_{i,j} = \begin{cases} 0, & i < j; \\ \binom{i}{j}\frac{i!}{j!}, & i \ge j, \end{cases}$$

satisfies that

(i) C_n is invertible and $C_n^{-1} = ((-1)^{i+j}c_{i,j})_{0\le i,j\le n}$;

(ii) for all $f \in \mathscr{T}_2^{(n)}$, $t > 0$,

$$(t^n f)^{(n)}(t) = \sum_{k=0}^n c_{n,k} t^k f^{(k)}(t),\ t^n f^{(n)}(t) = \sum_{k=0}^n (-1)^{k+n} c_{n,k}(t^k f)^{(k)}(t).$$

Next, we give some identities about holomorphic Fourier transform $\mathscr{F}$ and higher derivative, which are used in the proof of Theorem 3.1.

Lemma 3.4. *For all $n \in \mathbb{N}$, $z \in \overline{C}^+ \backslash \{0\}$, and $f \in \mathscr{T}_2^{(n)}$,*

$$(-1)^k z^k [\mathscr{F}(f)]^{(k)}(z) = \sum_{j=0}^{k} c_{k,j} \mathscr{F}(t^j f^{(j)})(z), \qquad k = 0, 1, \ldots, n;$$

$$(-1)^k \mathscr{F}(t^k f^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} z^j [\mathscr{F}(f)]^{(j)}(z), \qquad k = 0, 1, \ldots, n,$$

where $c_{k,j}$ is a constant defined by Lemma 3.3.

Proof. Let j,k be nonnegative integers such that $0 \le j \le k \le n$. Taking $h = t^k f$ and integrating by parts k times, we have $\mathscr{F}(h^{(k)})(z) = (-iz)^k \mathscr{F}(h)(z) - \sum_{j=0}^{k-1} (-iz)^{k-1-j} h^{(j)}(0)$. By applying Leibniz's rule of derivation in $(t^k f)^{(j)}$, we have $(t^k f)^{(j)}(0) = \lim_{t \to 0^+} (t^k f)^{(j)}(t) = 0$, $j = 0, 1, \ldots, k-1$. Thus

$$\mathscr{F}((t^k f)^{(k)})(z) = (-iz)^k \mathscr{F}(t^k f)(z) = (-1)^k z^k [\mathscr{F}(f)]^{(k)}(z).$$

and

$$(-1)^k z^k [\mathscr{F}(f)]^{(k)}(z) = \mathscr{F}((t^k f)^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} \mathscr{F}(t^j f^{(j)})(z),$$

where the second equality is based on Lemma 3.3. By applying the invertibility of the matrix defined by Lemma 3.3, we have $\mathscr{F}(t^j f^{(j)})(z) = \sum_{k=0}^{j} (-1)^j c_{j,k} z^k [\mathscr{F}(f)]^{(k)}(z)$. Interchange the position of j and k, we derive $(-1)^k \mathscr{F}(t^k f^{(k)})(z) = \sum_{j=0}^{k} c_{k,j} z^j [\mathscr{F}(f)]^{(j)}(z)$. □

Using the above lemma, now we can finish the proof of Theorem 3.1.

Proof. If $f \in \mathscr{T}_2^{(n)}$, then, for $0 \le k \le n$, it is easy to see that $t^k f^{(k)} \in L^2(\mathbb{R}^+)$. By applying Lemma 3.4 and the classical Paley-Wiener theorem, we derive that $z^k [\mathscr{F}(f)]^{(k)} \in H_2$. Thus $\mathscr{F}(f) \in H_2^{(n)}$.

Conversely, if $F \in H_2^{(n)} \subseteq H_2$, then, according to the classical Paley-Wiener theorem, there exists a function $f \in L^2(\mathbb{R}^+)$ such that $F = \mathscr{F}(f)$. All we need is to prove that $f \in \mathscr{T}_2^{(n)}$. To this end, we consider the function $G(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j F^{(j)}(z)$, $z \in \mathbb{C}^+$. Since $F \in H_2^{(n)}$, we have $G \in H_2$. Therefore, there exists a function $g \in L^2(\mathbb{R}^+)$ such that $G = \mathscr{F}(g)$. In addition, we define another function

$$h(s) = -\frac{1}{(n-1)!} \int_s^{\infty} (s-t)^{n-1} \frac{g(t)}{t^n} dt.$$

Note that $\frac{g(t)}{t^n} \in L^2(t^n)$. It is clear that it belongs to $\mathscr{T}_2^{(n)}$. By applying Lemma 3.1, we derive $g = t^n h^{(n)}$. According to Lemma 3.4, we have

$$G(z) = \mathscr{F}(g)(z) = \mathscr{F}(t^n h^{(n)})(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j [\mathscr{F}(h)]^{(j)}(z), \qquad z \in \mathbb{C}^+.$$

It follows from $F = \mathscr{F}(f)$ that $(-1)^n [z^n \mathscr{F}(h-f)]^{(n)}(z) = (-1)^n \sum_{j=0}^{n} c_{n,j} z^j [\mathscr{F}(h-f)]^{(j)}(z) = 0$, which implies that $z^n \mathscr{F}(h-f) = P_n(z)$, where $P_n(z)$ denotes the polynomial with its degree less than or equal to $n-1$. In view of $P_n(z) z^{-n} = \mathscr{F}(h-f) \in H_2$, we have $\mathscr{F}(h-f) = 0$. By

the injectivity of $\mathscr{F}$, we derive $f = h \in \mathscr{T}_2^{(n)}$. Therefore, $\mathscr{F}(\mathscr{T}_2^{(n)}) = H_2^{(n)}$. Taking $f, g \in \mathscr{T}_2^{(n)}$, we have $F = \mathscr{F}(f)$, $G = \mathscr{F}(g)$ in $H_2^{(n)}$. By Lemma 3.2, we have

$$\begin{aligned}\langle f,g\rangle_{\mathscr{T}_2^{(n)}} = \langle (t^n f)^{(n)}, (t^n g)^{(n)}\rangle_2 &= \langle \mathscr{F}[(t^n f)^{(n)}], \mathscr{F}[(t^n g)^{(n)}]\rangle_{H_2} \\ &= \langle (-1)^n z^n [\mathscr{F}(f)]^{(n)}, (-1)^n z^n [\mathscr{F}(g)]^{(n)}\rangle_{H_2} \\ &= \langle F, G\rangle_{H_2^{(n)}}.\end{aligned}$$

□

Note that the Cayley transform $\Phi(\lambda) = i\frac{1-\lambda}{1+\lambda}$, $\lambda \in \mathbb{D}$ maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{C}^+$ so that we could deal with Hardy-Sobolev spaces on $\mathbb{D}$ instead on the upper half-plane according to the following proposition.

Proposition 3.1. *Let F be an holomorphic function on $\mathbb{C}^+$ and $F_{\mathbb{D}}$ be the function on the unit disc given by $F_{\mathbb{D}}(\lambda) = F\left(i\frac{1-\lambda}{1+\lambda}\right)$, $\lambda \in \mathbb{D}$. Then $F \in H_2^{(1)}$ if and only if $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D})$ and $F'_{\mathbb{D}} \in (1-\lambda)^{-1}H_2(\mathbb{D})$.*

Proof. The condition $F \in H_2^{(1)}$ is equivalent to $F \in H_2(\mathbb{C}^+)$ and $zF' \in H_2(\mathbb{C}^+)$ by the definition of Hardy-Sobolev spaces. Note that $F \in H_2(\mathbb{C}^+)$ if and only if $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D})$ [20, Theorem 13.17], it follows that $zF' \in H_2(\mathbb{C}^+)$ is equivalent to

$$i\frac{1-\lambda}{1+\lambda}F'\left(i\frac{1-\lambda}{1+\lambda}\right) \in (1+\lambda)H_2(\mathbb{D}). \tag{3.1}$$

In addition, we have

$$F'_{\mathbb{D}}(\lambda) = \frac{-2i}{(1+\lambda)^2}F'\left(i\frac{1-\lambda}{1+\lambda}\right). \tag{3.2}$$

Combining formula (3.1) and (3.2), we obtain $F'_{\mathbb{D}}(\lambda) \in (1-\lambda)^{-1}H_2(\mathbb{D})$. □

Proposition 3.1 can be extended for $n > 1$ in the same way so that it offers a tool to investigate the function in Hardy-Sobolev spaces. We give another proposition of functions about Hardy-Sobolev spaces to end this section.

Proposition 3.2. *If $F \in H_2^{(n)}$, then $z^{k-1}F^{(k-1)}$, $k = 1,2,\ldots,n$ can be extended continuously at all nonzero point on the real axis and the point at infinity is included.*

Proof. Let $F \in H_2^{(1)}$. By Proposition 3.1, we have $F_{\mathbb{D}} \in (1+\lambda)H_2(\mathbb{D}) \subseteq H_2(\mathbb{D})$ and $(1-\lambda)F'_{\mathbb{D}} \in H_2(\mathbb{D})$. Note that

$$[(1-\lambda)F_{\mathbb{D}}(\lambda)]' = -F_{\mathbb{D}}(\lambda) + (1-\lambda)F'_{\mathbb{D}}(\lambda) \subseteq H_2(\mathbb{D}) \subseteq H_1(\mathbb{D}).$$

Then, function $(1-\lambda)F_{\mathbb{D}}(\lambda)$ can be extended continuously on the unit circle, that is, $F_{\mathbb{D}}$ extends by continuity to the unit circle except possibly at 1. Since the Cayley transform maps 1 to 0 and -1 to ∞, the conclusion on the extensibility of F by continuity follows. Now, we take $F \in H_2^{(n)}$, $n > 1$. In view of

$$z\frac{d}{dz}\left(z^{k-1}F^{(k-1)}(z)\right) = (k-1)z^{k-1}F^{(k-1)}(z) + z^k F^k(z), \qquad z \in \mathbb{C}^+,$$

we have $z^{k-1}F^{(k-1)}(z)$, which belongs to $H_2^{(1)}$ for $1 \le k \le n$. Thus the continuity on the real axis of functions $z^{k-1}F^{(k-1)}$, $k = 1,2,\ldots,n$ follows by the previous discussion. □

4. Reproducing Kernel of Hardy-Sobolev spaces

By applying Theorem 3.1, we can determine the integral form of reproducing kernels about Hardy-Sobolev spaces.

Theorem 4.1. *Let n be a positive integer. Then the function K_n defined on $\mathbb{C}^+ \times \mathbb{C}^+$ by*

$$K_n(z,w) = \frac{1}{((n-1)!)^2}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{i}{yz-x\overline{w}}dxdy, \qquad z,w\in\mathbb{C}^+,$$

is the reproducing kernel of $H_2^{(n)}$, i.e., if $K_{n,w}(z) = K_n(z,w)$, then $K_{n,w}\in H_2^{(n)}$ and $F(w) = \langle F, K_{n,w}\rangle_{H_2^{(n)}}$ for all $F\in H_2^{(n)}$.

Proof. Take $w\in\mathbb{C}^+$ and $n\in\mathbb{N}^*$, and let

$$\varphi_{w,n}(t) = \frac{1}{t^n}\int_0^1 \frac{(1-x)^{n-1}}{(n-1)!}e^{-i\overline{w}tx}dx, \qquad t>0.$$

Observe that $\varphi_{w,n}\in L^2(t^n)$. In fact,

$$\left(\int_0^\infty |\varphi_{w,n}(t)|^2 t^{2n}dt\right)^{\frac{1}{2}} \le \left(\int_0^\infty\left(\int_0^1\frac{(1-x)^{n-1}}{(n-1)!}e^{-(\Im w)tx}dx\right)^2 dt\right)^{\frac{1}{2}}$$
$$\le \int_0^1\frac{(1-x)^{n-1}}{(n-1)!}\left(\int_0^\infty e^{-2(\Im w)tx}\right)^{\frac{1}{2}}dx = \frac{\sqrt{\frac{\pi}{2}}}{\Gamma(n+\frac{1}{2})\sqrt{\Im w}}.$$

Hence, $g_{w,n}(t) = W^{-n}\varphi_{w,n}(t)\in\mathscr{T}_2^{(n)}$ and

$$g_{w,n}(t) = \frac{1}{((n-1)!)^2}\int_t^\infty\frac{(s-t)^{n-1}}{s^n}\int_0^1(1-x)^{n-1}e^{-i\overline{w}sx}dxds.$$

By Lemma 3.1, we have

$$g_{w,n}^{(n)}(t) = (-1)^n\varphi_{w,n}(t) = \frac{(-1)^n}{t^n}\int_0^1\frac{(1-x)^{n-1}}{(n-1)!}e^{-i\overline{w}tx}dx = \frac{(-1)^n}{t^{2n}}\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}e^{-i\overline{w}s}ds.$$

Taking $F\in H_2^{(n)}$, we see by Theorem 3.1 that there exists a function $f\in\mathscr{T}_2^{(n)}$ such that $F = \mathscr{F}(f)$. By applying Fubini's theorem, we have, for all $f\in\mathscr{T}_2^{(n)}$,

$$\begin{aligned}F(w) = \mathscr{F}(f)(w) = \int_0^\infty f(t)e^{iwt}dt &= (-1)^n\int_0^\infty\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}f^{(n)}(s)ds\,e^{iwt}dt\\ &= \int_0^\infty f^{(n)}(t)\int_0^t(-1)^n\frac{(t-s)^{n-1}}{(n-1)!}e^{iws}ds\\ &= \int_0^\infty f^{(n)}(t)t^{2n}\overline{g_{w,n}^{(n)}(t)}dt\\ &= \langle f, g_{w,n}\rangle_{\mathscr{T}_2^{(n)}}.\end{aligned}$$

On the other hand, using the change of variables $s = \frac{t}{y}$, we have

$$g_{w,n}(t) = \int_0^1\int_0^1\frac{(1-y)^{n-1}(1-x)^{n-1}}{y[(n-1)!]^2}e^{\frac{-i\overline{w}tx}{y}}dxdy.$$

For $z \in \mathbb{C}^+$, we see that there exists

$$\begin{aligned}\mathscr{F}(g_{w,n})(z) &= \int_0^1 \int_0^1 \frac{(1-y)^{n-1}(1-x)^{n-1}}{y[(n-1)!]^2} \mathscr{F}(e^{\frac{-i\overline{w}tx}{y}})(z)dxdy \\ &= \int_0^1 \int_0^1 \frac{(1-y)^{n-1}}{(n-1)!} \frac{(1-x)^{n-1}}{(n-1)!} \frac{i}{yz-x\overline{w}} dxdy \\ &= K_{n,w}(z).\end{aligned}$$

It follows from Theorem 3.1 that $F(w) = \langle f, g_{w,n} \rangle_{\mathscr{T}_2^{(n)}} = \langle F, K_{n,w} \rangle_{H_2^{(n)}}$, $w \in \mathbb{C}^+$. □

In the rest of this section, we need to estimate the norm of the reproducing kernel, which is important for the boundedness of weighted composition operators in Section 5. Before we give the estimation, we first prove the following key lemma.

Lemma 4.1. *For $\theta \in (0,\pi)$, let $I(\theta) = \int_0^1 \frac{sin\theta}{t^2+1-2tcos2\theta} dt$. Then*

$$I(\theta) = \begin{cases} \frac{1}{2}, & \theta = \frac{\pi}{2}; \\ \frac{\pi-2\theta}{4cos\theta}, & \theta \neq \frac{\pi}{2}. \end{cases}$$

and $\frac{1}{2} \leq I(\theta) \leq \frac{\pi}{4}$.

Proof. A straightforward calculation shows that $I(\frac{\pi}{2}) = \int_0^1 \frac{dt}{(t+1)^2} = \frac{1}{2}$. For $0 < \theta < \frac{\pi}{2}$, we have

$$\begin{aligned} I(\theta) &= \int_0^1 \frac{sin\theta}{(t-cos2\theta)^2 + sin^2 2\theta} dt \\ &= \frac{1}{2cos\theta} \int_0^{\frac{1}{sin2\theta}} \frac{dx}{\left(x - \frac{cos2\theta}{sin2\theta}\right)^2 + 1} \\ &= \frac{1}{2cos\theta} \int_{-\frac{cos2\theta}{sin2\theta}}^{\frac{sin\theta}{cos\theta}} \frac{dy}{y^2+1} = \frac{\pi - 2\theta}{4cos\theta}. \end{aligned}$$

Note that if $t \in (0, \frac{\pi}{2}), 1 \leq \frac{t}{sint} \leq \frac{\pi}{2}$, we have $I(t) = \frac{t}{2sint} \in \left[\frac{1}{2}, \frac{\pi}{4}\right]$ with changing the variables $t = \frac{\pi}{2} - \theta$. Since the case that $\frac{\pi}{2} < \theta < \pi$ is similar to the case $0 < \theta < \frac{\pi}{2}$, we omit the proof here. □

The following Theorem 4.2 gives the estimation of the reproducing kernel.

Theorem 4.2. *Let n be the positive integer. Then, for $z \in \mathbb{C}^+$,*

$$\frac{1}{(n-1)!\sqrt{2n-1}} \frac{1}{\sqrt{|z|}} \leq \|K_{n,z}\|_{H_2^{(n)}} \leq \frac{\sqrt{\pi}}{(n-1)!\sqrt{n}} \frac{1}{\sqrt{|z|}}.$$

Proof. For $z = |z|(cos\theta + isin\theta) \in \mathbb{C}^+$, we have

$$\begin{aligned} \frac{i}{yz - x\bar{z}} &= \frac{i}{|z|} \frac{1}{y(cos\theta + isin\theta) - x(cos\theta - isin\theta)} \\ &= \frac{i}{|z|} \frac{1}{(y-x)cos\theta + i(y+x)sin\theta} \\ &= \frac{1}{|z|} \frac{(y+x)sin\theta - i(y-x)cos\theta}{x^2 + y^2 - 2xycos2\theta}. \end{aligned}$$

Since the norm is always real, we derive by using the symmetry of the domain of integration that

$$\begin{aligned}\|K_{n,z}\|^2_{H_2^{(n)}} &= \langle K_{n,z}, K_{n,z}\rangle_{H_2^{(n)}} = K_{n,z}(z)\\ &= \frac{1}{[(n-1)!]^2}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{i}{yz-x\bar{z}}dxdy\\ &= \frac{1}{[(n-1)!]^2}\frac{1}{|z|}\int_0^1\int_0^1 (1-y)^{n-1}(1-x)^{n-1}\frac{(x+y)sin\theta}{x^2+y^2-2xycos2\theta}dxdy\\ &= \frac{2}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1+t)(1-yt)^{n-1}(1-y)^{n-1}sin\theta}{t^2+1-2tcos2\theta}dtdy.\end{aligned}$$

A straightforward calculation shows that $(1-y)^{n-1} \le (1+t)(1-yt)^{n-1} \le 2$ for $t, y \in (0,1)$. Hence

$$\|K_{n,z}\|^2_{H_2^{(n)}} \le \frac{4}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1-y)^{n-1}sin\theta}{t^2+1-2tcos2\theta}dtdy \le \frac{\pi}{[(n-1)!]^2 n}\frac{1}{|z|}.$$

On the other hand, we have

$$\begin{aligned}\|K_{n,z}\|^2_{H_2^{(n)}} &\ge \frac{2}{[(n-1)!]^2|z|}\int_0^1\int_0^1 \frac{(1-y)^{2n-2}sin\theta}{t^2+1-2tcos2\theta}dtdy\\ &= \frac{2I(\theta)}{[(n-1)!]^2(2n-1)|z|} \ge \frac{1}{[(n-1)!]^2(2n-1)}\frac{1}{|z|}.\end{aligned}$$

This completes the proof. □

5. Operator Theory

In this section, we are in a position to consider the multiplication operator and the weighted composition operator on $H_2^{(n)}$, Hardy-Sobolev spaces . We first study the spectrum of multiplication operators and list some facts.

Lemma 5.1. *If $\varphi \in \mathscr{M}_n$, then M_φ is a bounded linear operator from $H_2^{(n)}$ to itself.*

Proof. Assume that $f_m \to 0$ in $H_2^{(n)}$ and $M_\varphi(f_m) \to g$ in $H_2^{(n)}$. Note that $H_2^{(n)}$ is a RKHS so that, for each $z \in \mathbb{C}^+$, point evaluation functional at z is bounded, which means that there exists a constant $M > 0$ such that, for all functions $f \in H_2^{(n)}$, $|f(z)| \le M\|f\|$. For each $z \in \mathbb{C}^+$, $g(z) = \lim_{m\to\infty} M_\varphi(f_m)(z) = \varphi(z)\lim_{m\to\infty} f_m(z) = 0$. Thus $g = 0$, and we complete the proof by applying the close graph theorem. □

From Lemma 5.1, we can define the norm on $\mathscr{M}_n$ by $\|\varphi\|_{\mathscr{M}_n} = \|M_\varphi\| = \sup_{\|f\|_{H_2^{(n)}}\le 1}\|\varphi f\|_{H_2^{(n)}}$.

Lemma 5.2. *If $\varphi \in \mathscr{M}_n$, then $M^*_\varphi(K_{n,w}) = \overline{\varphi(w)}K_{n,w}$, $w \in \mathbb{C}^+$.*

Proof. For all $w \in \mathbb{C}^+$, if $f \in H_2^n$, then

$$\langle f, M^*_\varphi(K_{n,w})\rangle = \langle M_\varphi(f), K_{n,w}\rangle = \langle \varphi f, K_{n,w}\rangle = \varphi(w)f(w) = \langle f, \overline{\varphi(w)}K_{n,w}\rangle.$$

Hence $M^*_\varphi(K_{n,w}) = \overline{\varphi(w)}K_{n,w}$. □

Lemma 5.3. *If $\varphi \in \mathscr{M}_n$, then $\varphi \in H^\infty$ and $\|\varphi\|_{H^\infty} \leq \|\varphi\|_{\mathscr{M}_n}$.*

Proof. Let $\varphi \in \mathscr{M}_n$. It follows from Lemma 5.1 that M_φ^* is a bounded linear operator on $H_2^{(n)}$. Therefore, the eigenvalues of M_φ^* are bounded and their modulus is less than or equal to the spectral radius of M_φ^*, and also no more than $\|M_\varphi^*\|$. Note that $\|M_\varphi\| = \|M_\varphi^*\|$. According to the Lemma 5.2, we derive that the values of φ are bounded and of modulus no more than $\|\varphi\|_{\mathscr{M}_n}$. □

Lemma 5.4. *If $\varphi \in H_\infty^{(n)}$, then $\varphi \in \mathscr{M}_n$.*

Proof. Note that if $\varphi \in H_\infty^{(n)}$, then there exists a constant M such that, for all $k \in \mathbb{N}$ and $0 \leq k \leq n$, $\sup_{z,k} |z^k \varphi^{(k)}| \leq M$. Hence, if $f \in H_2^{(n)}$, then

$$\|\varphi f\|_{H_2^{(n)}} = \|z^n (\varphi f)^{(n)}\|_{H_2} = \|z^n \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)} f^{(n-k)}\|_{H_2} \leq M \sum_{k=0}^{n} \binom{n}{k} \|z^{n-k} f^{(n-k)}\|_{H_2}.$$

The last formula is finite due to the definition of the spaces $H_2^{(n)}$. This completes the proof. □

Next, we characterize the spectrum of multiplication operator M_φ.

Theorem 5.1. *For any $\varphi \in \mathscr{M}_n$, the spectrum of the multiplication operator $M_\varphi : H_2^{(n)} \to H_2^{(n)}$ is given by $\sigma(M_\varphi) = \overline{\varphi(\mathbb{C}^+)}$, which is the closure of the range of φ in the complex plane.*

Before we prove this theorem, we need the following proposition about $\mathscr{M}_n$.

Proposition 5.1. *If $\varphi \in \mathscr{M}_n$ and there exists a positive consant δ such that $|\varphi(z)| \geq \delta$ for all $z \in \mathbb{C}^+$, then $\frac{1}{\varphi} \in \mathscr{M}_n$.*

We use the following Lemma 5.5 to prove Proposition 5.1.

Lemma 5.5. *[7] Let N be a positive integer, f and g be N times differentiable on the open interval I, and g be non-vanishing on I. Then $D^N(\frac{f}{g}) = \frac{(-1)^N}{g^{N+1}} \sum_{k=0}^{N} (-1)^k \binom{N+1}{k} g^k D^N(g^{N-k} f)$. where $D^n = \frac{d^n}{dx^n}$, $n \geq 0$ is the n-th order differential operator.*

First, we prove Proposition 5.1.

Proof. We need to show that $\|z^n \left(\frac{f}{\varphi}\right)^{(n)}\|_{H_2} < \infty$ for all $f \in H_2^{(n)}$. If $\varphi \in \mathscr{M}_n$, then each function $\varphi^{(n-k)} f$ belongs to $H_2^{(n)}$, which means that $z^n D^n(\varphi^{(n-k)} f) \in H_2$ for all integer $0 \leq k \leq n$. By Lemma 5.5, we derive $\|\frac{f}{\varphi}\|_{H_2^{(n)}} = \|z^n \left(\frac{f}{\varphi}\right)^{(n)}\|_{H_2} = \|\frac{(-1)^n}{\varphi^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} \varphi^k z^n D^n(\varphi^{(n-k)} f)\|_{H_2}$. The last equation is finite since φ and $\frac{1}{\varphi}$ are both bounded holomorphic functions on $\mathbb{C}^+$. Thus $\frac{f}{\varphi} \in H_2^{(n)}$ whenever $f \in H_2^{(n)}$. □

Now we are in a position to prove Theorem 5.1.

Proof. For any $w \in \mathbb{C}^+$, it is easy to see that $\overline{\varphi(w)}$ is an eigenvalue of the operator M_φ^* according to the Lemma 5.2. Thus $\overline{\varphi(w)} \in \sigma(M_\varphi^*)$, which implies that $\varphi(w) \in \sigma(M_\varphi)$. Then we derive $\varphi(\mathbb{C}^+) \subseteq \sigma(M_\varphi)$. If $\lambda \in \mathbb{C} - \varphi(\mathbb{C}^+)$, then there exists a positive constant δ such that $|\lambda - \varphi(w)| \geq \delta$ for all $w \in \mathbb{C}^+$. Since $\lambda - \varphi \in \mathscr{M}_n$ due to the linear property of $\mathscr{M}_n$, we see that

$\psi = \frac{1}{\lambda - \varphi}$ is also the multiplier of $H_2^{(n)}$ by Proposition 5.1. It is easy to check that $M_\psi(\lambda I - M_\varphi) = (\lambda I - M_\varphi)M_\psi = I$. Hence, $\lambda I - M_\varphi$ is invertible, so $\lambda \notin \sigma(M_\varphi)$, which implies that $\sigma(M_\varphi) \subseteq \varphi(\mathbb{C}^+)$. Therefore, we conclude $\sigma(M_\varphi) = \varphi(\mathbb{C}^+)$. Taking closure on both sides, we complete the proof immediately. □

Corollary 5.1. *If $\varphi \in \mathscr{M}_n$, then $r(M_\varphi) = \|\varphi\|_\infty$, where $r(\cdot)$ denotes the spectral radius of operator.*

In the rest of this section, we discuss the boundedness of weighted composition operators on Hardy-Sobolev spaces. Recall that the notation $\mathscr{S}$ denotes the class of all analytic self-maps on $\mathbb{C}^+$. The following Julia-Carathéodory theorem for the upper half-plane was established in [18].

Lemma 5.6. *(Julia-Carathéodory) For $\varphi \in \mathscr{S}$, the following statement are equivalent:*
(a) $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$;
(b) $\sup_{z\in\mathbb{C}^+} \frac{\mathrm{Im} z}{\mathrm{Im}\varphi(z)} < \infty$;
(c) $\limsup_{z\to\infty} \frac{\mathrm{Im} z}{\mathrm{Im}\varphi(z)} < \infty$.
Moreover, the quantities in (b) and (c) are both equal to $\varphi'(\infty)$.

Next, we present a property of analytic self-map $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$.

Lemma 5.7. *If $\varphi \in \mathscr{S}$ and satisfies $\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty$, then $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.*

Proof. Let γ be the Caley transformation $\gamma(\lambda) = i\frac{1+\lambda}{1-\lambda}$, $\lambda \in \mathbb{D}$, which conformally maps $\mathbb{D}$ onto $\mathbb{C}^+$. Given $\varphi \in \mathscr{S}$, let $\phi = \gamma^{-1} \circ \varphi \circ \gamma$. Then, for $\lambda \in \mathbb{D}$ and $z = \gamma(\lambda) \in \mathbb{C}^+$, we have

$$\frac{z}{\varphi(z)} = \frac{1-\phi(\lambda)}{1+\phi(\lambda)} \frac{1+\lambda}{1-\lambda}.$$

Let $M = \sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|}$. Since $M < \infty$, we have $\varphi(\infty) = \infty$ and $\lim_{\lambda\to 1} \phi(\lambda) = 1$. In view of

$$\left|\frac{1-\phi(\lambda)}{1-\lambda}\right| \le M \left|\frac{1+\phi(\lambda)}{1+\lambda}\right|,$$

we derive $\limsup_{\lambda\to 1} |\frac{1-\phi(\lambda)}{1-\lambda}| < \infty$, so $\phi'(1) < \infty$. By the formula (2.18) in [18, p. 3178], we know that $\varphi'(\infty) < \infty$. Therefore, the proof of Lemma 5.7 follows. □

Lemma 5.8. *If $\psi \in \mathscr{M}_n$, then $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ if and only if $C_{\psi,\varphi} H_2^{(n)} \subseteq H_2^{(n)}$.*

Proof. If $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$, then $\|C_{\psi,\varphi} f\|_{H_2^{(n)}} \le \|C_{\psi,\varphi}\| \|f\|_{H_2^{(n)}}$. If $f \in H_2^{(n)}$, we have $C_{\psi,\varphi} f \in H_2^{(n)}$. On the other hand, we assume that $f_m \to 0$ and $C_{\psi,\varphi} f_m \to g$. Then, for each $z \in \mathbb{C}^+$, $g(z) = \lim_{m\to\infty} C_{\psi,\varphi} f_m(z) = \psi(z) \lim_{m\to\infty} f(\varphi(z))$. Since $H_2^{(n)}$ is a RKHS, we have $g(z) = 0$ for all $z \in \mathbb{C}^+$ and then $g = 0$. From the closed graph principle, we obtain that $C_{\psi,\varphi}$ is bounded. □

Next, we investigate some properties of the weighted composition operator $C_{\psi,\varphi}$ on $H_2^{(n)}$ as follows.

Theorem 5.2. *If ψ is in $\mathscr{M}_n$ satisfying that there exists a positive consant δ such that $|\psi(z)| \geq \delta$ for all $z \in \mathbb{C}^+$ and $C_{\psi,\varphi}$ is a bounded composition operator on $H_2^{(n)}$ for $n \geq 1$, then the self-map φ satisfies $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$.*

Proof. Using the property of reproducing kernel, we can write

$$\langle C^*_{\psi,\varphi} K_{n,z}, f\rangle = \langle K_{n,z}, \psi f\circ\varphi\rangle = \overline{\psi(z) f\circ\varphi(z)} = \langle \overline{\psi(z)} K_{n,\varphi(z)}, f\rangle, \quad f \in H_2^{(n)},$$

and then $C^*_{\psi,\varphi} K_{n,z} = \overline{\psi(z)} K_{n,\varphi(z)}$, $z \in \mathbb{C}^+$. Hence,$\sup_{z\in\mathbb{C}^+} |\psi(z)| \frac{\|K_{n,\varphi(z)}\|_{H_2^{(n)}}}{\|K_{n,z}\|_{H_2^{(n)}}} \leq \|C^*_{\psi,\varphi}\| = \|C_{\psi,\varphi}\|$. By Theorem 4.2, we derive

$$B := |\psi(z)|^2 \frac{\|K_{n,\varphi(z)}\|^2_{H_2^{(n)}}}{\|K_{n,z}\|^2_{H_2^{(n)}}} \geq \frac{|\psi(z)|^2 \frac{1}{[(n-1)!]^2(2n-1)|\varphi(z)|}}{\frac{\pi}{[(n-1)!]^2 n|z|}} = \frac{n|\psi(z)|^2}{(2n-1)\pi} \frac{|z|}{|\varphi(z)|}.$$

In view of $\psi^{-1} \in H^\infty$ and $B < \infty$ for all $z \in \mathbb{C}^+$, we have $\sup_{z\in\mathbb{C}^+} \frac{|z|}{|\varphi(z)|} < \infty$. Thus, by Lemma 5.7, $\varphi(\infty) = \infty$ and $\varphi'(\infty) < \infty$. □

We also give a sufficient condition for boundedness of weighted composition operators.

Theorem 5.3. *If$\psi \in H_\infty^{(n)}$, $\varphi \in \mathscr{S}$ satisfy $\varphi'(\infty) < \infty$ and $\sup_{z\in\mathbb{C}^+} |z^k \frac{\varphi^{(k)}(z)}{\varphi(z)}| < \infty$, $k = 1,2,\ldots,n$, then the weighted composition operator $C_{\psi,\varphi} : H_2^{(n)} \to H_2^{(n)}$ is bounded.*

Proof. By Lemma 5.6, we need to show that if f is an arbitrary function in $H_2^{(n)}$, then $\psi f\circ\varphi \in H_2^{(n)}$. Faà di Bruno shows that $(f\circ\varphi)^{(n)} = \sum_{k=1}^n c_k (f^{(k)}\circ\varphi)(\varphi')^{m_{k_1}}(\varphi'')^{m_{k_2}}\ldots(\varphi^{(n)})^{m_{k_n}}$, where c_k, $k = 1,2,\ldots,n$ and m_{i_j}, $i,j = 1,2\ldots,n$ are nonnegative integers and satisfy that, for $k = 1,2,\ldots,n$, $\sum_{p=1}^n p m_{k_p} = n$ and $\sum_{p=1}^n m_{k_p} = k$. It is easy to see that there exists a $M \geq 0$ such that, for $k = 0,1,\ldots,n$, $\sup_{z\in\mathbb{C}^+} |z^k \psi^{(k)}| \leq M$. Thus, we have the following formula:

$$\begin{aligned}
&\|z^n(\psi f\circ\varphi)^{(n)}\|_{H_2}\\
=&\|z^n \sum_{k=0}^n \binom{n}{k} \psi^{(n-k)} \sum_{l=1}^k c_l (f^{(l)}\circ\varphi)(\varphi')^{m_{l_1}}\ldots(\varphi^{(k)})^{m_{l_k}}\|_{H_2}\\
\leq&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l \|z^{n-k}\psi^{(n-k)} z^k (f^{(l)}\circ\varphi)(z)(\varphi'(z))^{m_{l_1}}\ldots(\varphi^{(k)}(z))^{m_{l_k}}\|_{H_2}\\
=&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l \|z^{n-k}\psi^{(n-k)}(f^{(l)}\circ\varphi)(z)\left(\frac{z\varphi'(z)}{\varphi(z)}\right)^{m_{l_1}}\ldots\left(\frac{z^k\varphi^{(k)}(z)}{\varphi(z)}\right)^{m_{l_k}}(\varphi(z))^l\|_{H_2}\\
\leq&\sum_{k=0}^n\sum_{l=1}^k \binom{n}{k} c_l M \prod_{j=1}^k \left\|\frac{z^j\varphi^{(j)}(z)}{\varphi(z)}\right\|_\infty^{m_{l_j}} \|(\varphi(z))^l (f^{(l)}\circ\varphi)(z)\|_{H_2}.
\end{aligned}$$

Not all analytic self-maps φ on $\mathbb{C}^+$ induce bounded composition operators on H_2, the necessary and sufficient condition for boundedness being $\varphi'(\infty) < \infty$ ([19]). Given that $z^l f^{(l)}(z) \in H_2$ for all $l = 1,2,\ldots,k$ and $\varphi'(\infty) < \infty$, it follows that C_φ is a bounded operator on H_2, that is,

$$\|C_\varphi(z^l f^{(l)}(z))\|_{H_2} = \|(z^l f^{(l)})\circ\varphi\|_{H_2} = \|(\varphi(z))^l (f^{(l)}\circ\varphi)(z)\|_{H_2} < \infty, \quad l = 1,2,\ldots,k,$$

so we show that $\|z^n(\psi f\circ\varphi)^{(n)}\|_{H_2}$ is finite and $\psi f\circ\varphi\in H_2^{(n)}$. The proof is completed. □

Moreover, for the richness of the properties of operators in Hardy-Sobolev spaces, we have a related conclusion based on [9, Theorem 7.1], which is also a main result in Hardy-Sobolev spaces. Applying the theorem above in the case of $H_2^{(n)}$, we have the following corollary.

Corollary 5.2. *Let $\varphi\in\mathscr{S}$ and $n\in\mathbb{N}$. Then the weighted composition operator is a bounded operator on $H_2^{(n)}$ if and only if there exists a $M\geq 0$ such that the kernel given by*

$$L_n(z,w)=\int_0^1\int_0^1(1-y)^{n-1}(1-x)^{n-1}i\left(\frac{M^2}{yz-x\overline{w}}-\frac{\overline{\psi(z)}\psi(w)}{y\varphi(z)-x\overline{\varphi(w)}}\right)dxdy,$$

is nonnegative.

Acknowledgments

This work was supported by the NSF of China (Grant No. 12326407, 12071155, and 82361168672). This work was also partly supported by The Science and Technology Development Fund, Macau SAR (File no. FDCT/0036/2021/AGJ) and University of Macau (File no. MYRG2022-00108-FST).

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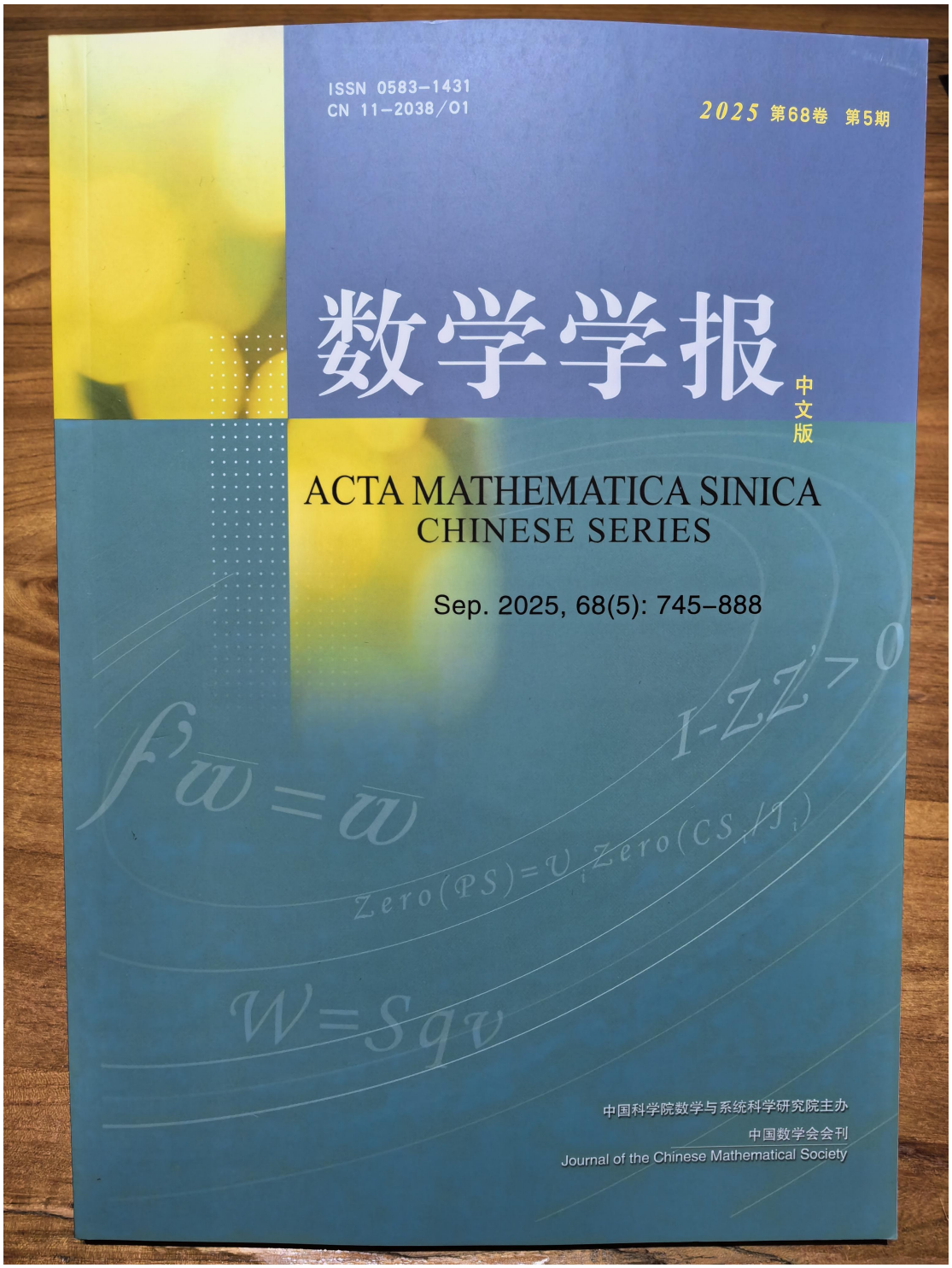
ISSN 0583-1431
CN 11-2038/O1
2025 第68卷 第5期
数学学报
中文版
ACTA MATHEMATICA SINICA
CHINESE SERIES
Sep. 2025, 68(5): 745–888
中国科学院数学与系统科学研究院主办
中国数学会会刊
Journal of the Chinese Mathematical Society

中国科学院科学出版基金资助

数学学报中文版

第 68 卷 第 5 期 2025 年 9 月

目 次

具有非负预定函数的广义 Kazdan–Warner 方程解的存在性 …… 杨帼怡 朱晓宝 (745)

关于限制型 4- 着色分拆的同余式 …… 马无瑕 陈永高 (755)

δ-BiHom-Jordan 李超三系的广义导子 …… 宋 硕 唐黎明 (765)

一类带有非 Lipschitz 系数的平均场倒向随机微分方程强解的存在唯一性 …… 郝 涛 (781)

配对设计中相对风险置信区间的构造 …… 贺 欢 何 笑 张丽平 田茂再 (799)

区间上平顶反双峰连续自映射的迭代根 …… 孙太祥 秦 斌 韩彩虹 (820)

F-Willmore 超曲面的谱特征 …… 杨登允 张金国 陶永芊 (831)

置换多项式中的一个计数问题 …… 李 娟 官欢欢 吴丹尧 (839)

一类挠性 Dirac 算子的 Einstein–Hilbert 作用 …… 王 剑 吴 彤 包开花 (847)

有限域上三角代数簇的有理点个数 …… 洪思奥 朱光艳 (857)

多复变数准凸映射推广的 Fekete–Szegö 不等式及其应用 …… 刘小松 李海绸 (872)

本刊英文版 Vol. 41(2025), Nos. 7 & 8 论文摘要 …… (i–viii)

第68卷第5期 数 学 学 报 中文版 Vol.68, No.5
2025年9月 ACTA MATHEMATICA SINICA, CHINESE SERIES Sep., 2025

DOI: 10.12386/A20250014 文献标识码: A

多复变数准凸映射推广的 Fekete–Szegö 不等式及其应用

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摘 要 本文给出复 Banach 单位球上和 $\mathbb{C}^n$ 中单位多圆柱上一类准凸映射 (含 $\mathbb{A}$ 型准凸映射与 $\mathbb{B}$ 型准凸映射) 推广的 Fekete–Szegö 不等式, 作为主要结果的应用同时建立了上述映射在相应域上相邻齐次展开式范数差的估计. 所得结果既能回到单复变数经典的结果又推广了多复变数一些已知结论.

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MR(2020) 主题分类 32A30, 32H02, 30C45
中图分类 O174.56

The Generalized Fekete and Szegö Inequality for Quasi-convex Mappings in Several Complex Variables and Its Applications

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收稿日期: 2024-10-21; 接受日期: 2025-03-07
基金项目: 国家自然科学基金资助项目 (11871257, 12071130, 12326407, 12071155) 与岭南师范学院资助项目 (LT2410), 广东省自然科学基金面上项目 (2025A1515011213)
通讯作者: 李海绸

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type $\mathbb{B}$) defined on the unit ball of complex Banach spaces and the unit polydisc in $\mathbb{C}^n$. We also establish the successive homogeneous expansions difference bounds for the above mappings defined on the corresponding domains as applications of the main results. These obtained results not only reduce to the classical result in one complex variable but also generalize some known results in several complex variables.

Keywords The generalized Fekete and Szegö inequality; successive homogeneous expansions difference bound; quasi-convex mapping; quasi-convex mappings of type $\mathbb{A}$; quasi-convex mappings of type $\mathbb{B}$; determinant
MR(2020) Subject Classification 32A30, 32H02, 30C45
Chinese Library Classification O174.56

1 引言

在单复变数理论中, 经典的 Fekete–Szegö 不等式发挥了奠基性作用, 尤其在估计单叶解析函数 Taylor 级数第三项系数问题上. 该估计与作为单复变几何函数论核心问题之一著名的 Bieberbach 猜想紧密相关.

定理 1.1 (见 [4]) 设 $\mu \in [0,1), f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ 是单位圆盘 U 上的单叶全纯函数, 则

$$|a_3 - \mu a_2^2| \leq 1 + 2e^{\left(-\frac{2\mu}{1-\mu}\right)},$$

且上述估计是精确的.

作为单叶解析函数最重要子类之一的双全纯凸函数, Keogh 与 Merkes 在 1969 年得到了如下 Fekete–Szegö 不等式精确的估计.

定理 1.2 (见 [10]) 设 $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ 是单位圆盘 U 上的双全纯凸函数, 则

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}, |\mu - 1|\right\}, \quad \mu \in \mathbb{C},$$

且上述估计是精确的.

一个令人感兴趣的问题是: 上述定理能否推广至多复变数. 不少重要工作表明 Fekete–Szegö 不等式对准凸映射在复 Banach 单位球上和 $\mathbb{C}^n$ 中单位多圆柱上 (含 $\mathbb{A}$ 型准凸映射与 $\mathbb{B}$ 型准凸映射) 的特定子类和双全纯映射的一些子类均成立, 不够遗憾的是, 这些结果一般在较强的限制条件下才成立 (参见文献 [1–3, 7–9, 15–17]). 因此很有必要将限制条件放宽或给出多复变数上述映射一些新的推广结论.

现回顾本文将要用到的一些记号和定义. X 表示具有范数 $\|\cdot\|$ 的复 Banach 空间, X^* 为 X 的对偶空间. E 是 X 的开单位球, ∂E 为其边界. U 记为 $\mathbb{C}$ 中的开单位圆盘, U^n 为 $\mathbb{C}^n$ 的开单位多圆柱. 类似地, ∂U^n 为 U^n 的边界, $(\partial U)^n$ 为 U^n 的特征边界, 且 $\overline{U^n}$ 为 U^n 的闭包. 符号 $'$ 表示转置. 对任意 $x \in X \setminus \{0\}$, 有

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

$H(E)$ 为 E 到 X 的全体全纯映射所成之集. 若 $f \in H(E)$, 则

$$f(y) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m f(x)((y-x)^m),$$

对 $x\in E$ 某邻域中所有 y 均成立, 其中 $D^mf(x)$ 是 f 在 x 的 m 阶 Fréchet 导数, 且对 $m\geq 1$, 有

$$D^mf(x)((y-x)^m)=D^mf(x)(\underbrace{y-x,\ldots,y-x}_{m}).$$

定义 1.3 (见 [18]) 设 $f:E\to X$ 是正规化局部双全纯映射, 且记

$$G_f(\alpha,\beta)=\frac{2\alpha}{T_u[(Df(\alpha u))^{-1}(f(\alpha u)-f(\beta u))]}-\frac{\alpha+\beta}{\alpha-\beta}.$$

若

$$\Re e\, G_f(\alpha,\beta)\geq 0,\quad u\in\partial E,\quad \alpha,\beta\in U,$$

则称 f 为 E 上的 $\mathbb{A}$ 型准凸映射.

记 $Q_{\mathbb{A}}(E)$ 为 E 上全体 $\mathbb{A}$ 型准凸映射所成之集.

定义 1.4 (见 [5]) 设 $f:E\to X$ 是正规化局部双全纯映射. 若

$$\Re e\,\{T_x[(Df(x))^{-1}(D^2f(x)(x^2)+Df(x)x)]\}\geq 0,\quad x\in E,$$

则称 f 为 E 上的 $\mathbb{B}$ 型准凸映射.

记 $Q_{\mathbb{B}}(E)$ 为 E 上全体 $\mathbb{B}$ 型准凸映射所成之集.

定义 1.5 (见 [18]) 设 $f:E\to X$ 是正规化局部双全纯映射. 若

$$\Re e\,\{T_x[(Df(x))^{-1}(f(x)-f(\xi x))]\}\geq 0,\quad x\in E,\quad \xi\in\overline{U}$$

则称 f 为 E 上的准凸映射.

记 $Q(E)$ 为 E 上全体准凸映射所成之集. 龚升 [5] 建立三类准凸映射如下包含关系, 即

$$Q_{\mathbb{A}}(E)=Q(E)\subset Q_{\mathbb{B}}(E). \tag{1.1}$$

2 一些引理

为获得后续节中的主要定理, 有必要建立下面引理.

类似文 [12, 引理3.1] 的证明, 易得下列引理 (略去细节).

引理 2.1 设 $f(x)=x+\sum_{m=2}^{\infty}\frac{D^mf(0)(x^m)}{m!}$ 是 E 上的正规化局部双全纯映射, 且 $g(x)=(Df(x))^{-1}(Df(x)x+D^2f(x)(x^2))$, 则对 $x\in E,\ m=3,4,\ldots$, 有

$$2\frac{D^2f(0)(x^2)}{2!}=\frac{D^2g(0)(x^2)}{2!}, \tag{2.1}$$

$$\begin{aligned}m(m-1)\frac{D^mf(0)(x^m)}{m!}=&\frac{D^mg(0)(x^m)}{m!}+2\frac{D^2f(0)(x,\frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!})}{2!}\\&+\cdots+(m-1)\frac{D^{m-1}f(0)(x^{m-2},\frac{D^2g(0)(x^2)}{2!})}{(m-1)!}.\end{aligned}$$

引理 2.2 设 $\nu=2,3,\ldots$, $f(x)=x+\sum_{m=2}^{\nu+1}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=\nu+2}^{\infty}\frac{D^mf(0)(x^m)}{m!}$ 是 E 上的正规化局部双全纯映射, 且 $g(x)=(Df(x))^{-1}(Df(x)x+D^2f(x)(x^2))$, 其中 $h:E\to\mathbb{C}$ 是全纯函数, 则对 $m=2,3,\ldots,\nu+1$, 有

$$m(m-1)\frac{D^mf(0)(x^m)}{m!}=m(m-1)x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}$$

$$= \frac{D^m g(0)(x^m)}{m!} + 2\frac{D^2 f(0)(x, \frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!})}{2!} + \cdots + (m-1)\frac{D^{m-1}f(0)(x^{m-2}, \frac{D^2 g(0)(x^2)}{2!})}{(m-1)!}, \tag{2.2}$$

$$\frac{D^m g(0)(x^m)}{m!} = x\psi_{m-1}(x), \tag{2.3}$$

其中

$$\psi_{m-1}(x) = \begin{vmatrix} 1 & 0 & \cdots & 2\cdot 1x(Dh(0)x) \\ 2Dh(0)x & 1 & \cdots & 3\cdot 2x\frac{D^2h(0)(x^2)}{2!} \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)\frac{D^{m-2}h(0)(x^{m-2})}{(m-2)!} & (m-2)\frac{D^{m-3}h(0)(x^{m-3})}{(m-3)!} & \cdots & m(m-1)x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!} \end{vmatrix}.$$

证明 由引理 2.1 易知 (2.2) 成立. 于是由 (2.2), 有

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 2Dh(0)x & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)\frac{D^{m-2}h(0)(x^{m-2})}{(m-2)!} & (m-2)\frac{D^{m-3}h(0)(x^{m-3})}{(m-3)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \frac{D^2g(0)(x^2)}{2!} \\ \frac{D^3g(0)(x^3)}{3!} \\ \vdots \\ \frac{D^mg(0)(x^m)}{m!} \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 1x(Dh(0)x) \\ 3\cdot 2x\frac{D^2h(0)(x^2)}{2!} \\ \vdots \\ m(m-1)x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!} \end{pmatrix}, \quad m = 2, 3, \ldots, \nu+1.$$

显见对 $m = 2, 3, \ldots, \nu+1$, 有

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 2Dh(0)x & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)\frac{D^{m-2}h(0)(x^{m-2})}{(m-2)!} & (m-2)\frac{D^{m-3}h(0)(x^{m-3})}{(m-3)!} & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & \frac{D^2g(0)(x^2)}{2!} \\ 0 & 1 & \cdots & \frac{D^3g(0)(x^3)}{3!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{D^mg(0)(x^m)}{m!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 2\cdot 1x(Dh(0)x) \\ 2Dh(0)x & 1 & \cdots & 3\cdot 2x\frac{D^2h(0)(x^2)}{2!} \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)\frac{D^{m-2}h(0)(x^{m-2})}{(m-2)!} & (m-2)\frac{D^{m-3}h(0)(x^{m-3})}{(m-3)!} & \cdots & m(m-1)x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!} \end{pmatrix}$$

从而由行列式的性质知 (2.3) 成立.

引理 2.3 设 $a_1=1$, $|b_{m-1}|\le 2$, $m=2,3,\ldots$, 其中每个 b_{m-1} 是复常数. 若

$$m(m-1)a_m=\sum_{k=1}^{m-1}ka_kb_{m-k},\quad m=2,3,\ldots, \tag{2.4}$$

则

$$a_m=\frac{1}{(m-1)!m!}\begin{vmatrix} 1\cdot 2 & 0 & \ldots & b_1\\ -2b_1 & 2\cdot 3 & \ldots & b_2\\ \vdots & \vdots & \ddots & \vdots\\ -2b_{m-2} & -b_{m-3} & \ldots & b_{m-1}\end{vmatrix},\quad m=2,3,\ldots, \tag{2.5}$$

且

$$|a_m|\le 1,\quad m=2,3,\ldots. \tag{2.6}$$

证明 由 (2.4) 得

$$\begin{pmatrix} 1\cdot 2 & 0 & \ldots & 0\\ -2b_1 & 2\cdot 3 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -2b_{m-2} & -3b_{m-3} & \ldots & (m-1)m\end{pmatrix}\begin{pmatrix} a_2\\ a_3\\ \vdots\\ a_m\end{pmatrix}=\begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_{m-1}\end{pmatrix},$$

与

$$\begin{pmatrix} 1\cdot 2 & 0 & \ldots & 0\\ -2b_1 & 2\cdot 3 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -2b_{m-2} & -3b_{m-3} & \ldots & (m-1)m\end{pmatrix}\begin{pmatrix} 1 & 0 & \ldots & a_2\\ 0 & 1 & \ldots & a_3\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & a_m\end{pmatrix}$$
$$=\begin{pmatrix} 1\cdot 2 & 0 & \ldots & b_1\\ -2b_1 & 2\cdot 3 & \ldots & b_2\\ \vdots & \vdots & \ddots & \vdots\\ -2b_{m-2} & -3b_{m-3} & \ldots & b_{m-1}\end{pmatrix}.$$

因此由行列式的性质知 (2.5) 成立.

另一方面, (2.4) 与引理 2.3 的假设蕴含

$$m(m-1)|a_m|\le 2\sum_{k=1}^{m-1}k|a_k|,\quad m=2,3,\ldots$$

成立. 定义如下非负整数数列 $\{A_m\}$

$$m(m-1)A_m=2\sum_{k=1}^{m-1}kA_k,\quad m=2,3,\ldots,$$

其中 $A_1=1$. 故由数学归纳法直接证得 $|a_m|\le A_m$. 再对下面行列式进行如下列变换: 第一列加上第 $(m-1)$ 列乘以 2 替代第一列, 第二列加上第 $(m-1)$ 列乘以 3 替代第二列, ..., 第 $(m-2)$

列加上第 $(m-1)$ 列乘以 $m-2$ 替代第 $(m-2)$ 列, 得

$$\begin{aligned}A_m &= \frac{1}{(m-1)!}\begin{vmatrix} 1\cdot 2 & 0 & 0 & \cdots & 2\\ -4 & 2\cdot 3 & 0 & \cdots & 2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -4 & -6 & -8 & (m-2)(m-1) & 2\\ -4 & -6 & -8 & \cdots & 2\end{vmatrix}\\ &= \frac{1}{(m-1)!m!}\begin{vmatrix} 2\cdot 3 & 6 & 8 & \cdots & 2\\ 0 & 3\cdot 4 & 8 & \cdots & 2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & (m-1)m & 2\\ 0 & 0 & 0 & \cdots & 1\cdot 2\end{vmatrix}\\ &= \frac{(m-1)!m!}{(m-1)!m!}\\ &= 1.\end{aligned}$$

最后由 (2.5) 知 $|a_m|\leq 1, m=2,3,\ldots$. 故所证结论成立.

利用 [13, 引理2.5–2.6, 推论2.3] 直接叙述下面引理 (略去证明).

引理 2.4 设 $f(z)=z+\sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$). 若

$$\frac{D^m f_p(0)(z^m)}{m!}=\sum_{l_1,l_2,\ldots,l_m=1}^{n}\frac{1}{m!}\,|a_{pl_1l_2\cdots l_m}|\,\mathrm{e}^{\mathrm{i}\frac{\theta_{pl_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}}z_{l_1}z_{l_2}\cdots z_{l_m},$$

其中 $\mathrm{i}=\sqrt{-1}$, $\theta_{pl_q}\in(-\pi,\pi]\,(p,q=1,2,\ldots,n)$, $a_{pl_1l_2\cdots l_m}=\frac{\partial^m f_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_m}}$, 则对 $m=2,3,\ldots$, 有

$$\sup_{\|z\|=1,\|w\|=1}\|D^m f(0)(z^{m-1},w)\|=\sup_{\|z\|=1}\|D^m f(0)(z^m)\|$$

与

$$\frac{\|D^m f(0)(z^m)\|}{m!}\leq\|z\|^m,$$

且上述估计是精确的.

3 复 Banach 单位球上准凸映射推广的 Fekete–Szegö 不等式及其应用

现给出本节的主要结果: 复 Banach 单位球上准凸映射推广的 Fekete–Szegö 不等式, 该定理将单复变数经典的 Fekete–Szegö 不等式推广至多复变数, 即定理 3.1 将单叶解析函数理论中著名的 Fekete–Szegö 不等式推广至复 Banach 空间单位球的情形.

定理 3.1 设 $\nu=2,3,\ldots, f(x)=x+\sum_{m=2}^{\nu+1}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=\nu+2}^{\infty}\frac{D^m f(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$ ($Q_{\mathbb{A}}(E)$ 或 $Q(E)$), 其中 $h: E\to\mathbb{C}$ 是全纯函数, 且 $h(0)=1$. 若 $\mu\in(-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty), m=2,\ldots,\nu$, 则

$$\left\|\frac{D^{m+1}f(0)(x^{m+1})}{(m+1)!}-\mu\frac{D^m f(0)(x^{m-1},\frac{D^2f(0)(x^2)}{2!})}{m!}\right\|\leq|1-\mu|\|x\|^{m+1},\quad x\in E,\ m=2,\ldots,\nu,$$

且上述估计是精确的.

证明 因 $f(x)=x+\sum_{m=2}^{\nu+1}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=\nu+2}^{\infty}\frac{D^mf(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$, 则

$$g(x)=(Df(x))^{-1}(Df(x)x+D^2f(x)(x^2)),$$

且 $\Re e(g(x))\geq 0, x\in E$. 固定 $x\in E\backslash\{0\}$, 定义

$$p(\lambda)=\begin{cases}\frac{T_x(g(\lambda x_0))}{\lambda}, & \lambda\in U\setminus\{0\},\\ 1, & \lambda=0,\end{cases}$$

其中 $x_0=\frac{x}{\|x\|}$. 则由 (2.3) 知 $\Re e(p(\lambda))>0, \lambda\in U$, 且

$$p(\lambda)=1+\sum_{m=2}^{\infty}b_{m-1}\lambda^{m-1}=1+\sum_{m=2}^{\nu+1}\psi_{m-1}(x)\lambda^{m-1}+\sum_{m=\nu+2}^{\infty}T_x\left(\frac{D^mg(0)(x_0^m)}{m!}\right)\lambda^{m-1},$$

比较上述等式两边系数, 得

$$b_{m-1}=\frac{D^{m-1}\psi_{m-1}(0)(x_0^{m-1})}{(m-1)!},\quad m=2,3,\ldots,\nu+1.$$

又记

$$a_{m+1}=\frac{D^mh(0)(x_0^m)}{m!},\quad m=2,3,\ldots,\nu.$$

由引理 2.2 推得

$$(m+1)ma_{m+1}=\sum_{k=1}^{m}ka_kb_{m+1-k},\quad m=2,3,\ldots,\nu.$$

注意到

$$|b_{m-1}|=\left|T_x\left(\frac{D^mg(0)(x_0^m)}{m!}\right)\right|\leq 2,\quad m=2,3,\ldots, \tag{3.1}$$

(见文 [6, 定理2.1.5]), 从而由 (2.6) 知

$$|a_m|\leq 1,\quad m=2,3,\ldots,\nu. \tag{3.2}$$

由 (3.1), (3.2) 与 [14, 引理1] 知

$$\begin{aligned}\left|\frac{2m}{m+1}a_ma_2-a_{m+1}\right|&=\left|\frac{m}{m+1}b_1a_m-\frac{1}{(m+1)m}\sum_{k=1}^{m}ka_kb_{m+1-k}\right|\\&=\left|\frac{m-1}{m+1}b_1a_m-\frac{1}{(m+1)m}\sum_{k=1}^{m-1}ka_kb_{m+1-k}\right|\\&=\left|\frac{1}{(m+1)m}b_1\sum_{k=1}^{m-1}ka_kb_{m-k}-\frac{1}{(m+1)m}\sum_{k=1}^{m-1}ka_kb_{m+1-k}\right|\\&=\left|\frac{1}{(m+1)m}\sum_{k=1}^{m-1}ka_k(b_1b_{m-k}-b_{m+1-k})\right|\\&\leq\frac{1}{(m+1)m}\sum_{k=1}^{m-1}k|a_k||b_1b_{m-k}-b_{m+1-k}|\\&\leq\frac{1}{(m+1)m}\sum_{k=1}^{m-1}2k\end{aligned}$$

$$= \frac{m-1}{m+1}, \quad m = 2, 3, \ldots, \nu. \tag{3.3}$$

因此对 $\mu \in (-\infty, \frac{2}{m+1}]$, 有

$$\begin{aligned}
&|a_{m+1} - \mu a_m a_2| \\
&= \left| \frac{1}{(m+1)m}(b_m + 2a_2 b_{m-1} + \cdots + (m-1)a_{m-1}b_2) + \left(\frac{2}{m+1} - \mu\right) a_m a_2 \right| \\
&\le \frac{1}{(m+1)m}(|b_{m-1}| + 2|a_2||b_{m-1}| + \cdots + (m-1)|a_{m-1}||b_1|) + \left(\frac{2}{m+1} - \mu\right)|a_m||a_2| \\
&\le \frac{1}{(m+1)m}(2 + 2\cdot 2 + \cdots + (m-1)\cdot 2) + \frac{2}{m+1} - \mu \\
&= 1 - \mu, \quad m = 2, 3, \ldots, \nu,
\end{aligned}$$

且对 $\mu \in [\frac{2m}{m+1}, +\infty)$, 由 (3.2) 与 (3.3) 知

$$\begin{aligned}
|a_{m+1} - \mu a_m a_2| &\le \left| a_{m+1} - \frac{2m}{m+1} a_m a_2 \right| + \left(\mu - \frac{2m}{m+1}\right)|a_m||a_2| \\
&\le \frac{m-1}{m+1} + \mu - \frac{2m}{m+1} \\
&= \mu - 1, \quad m = 2, 3, \ldots, \nu.
\end{aligned}$$

又注意到

$$\begin{aligned}
a_{m+1} &= T_x\left(\frac{D^{m+1}f(0)(x_0^{m+1})}{(m+1)!}\right) = \frac{D^m h(0)(x_0^m)}{m!}, \\
a_m a_2 &= T_x\left(\frac{D^m f(0)(x_0^{m-1}, \frac{D^2 f(0)(x_0^2)}{2!})}{m!}\right) \\
&= Dh(0)x_0 T_x\left(\frac{D^m f(0)(x_0^m)}{m!}\right) \\
&= \frac{D^{m-1}h(0)(x_0^{m-1})}{(m-1)!} Dh(0)x_0,
\end{aligned}$$

故有

$$\begin{aligned}
&\left\| \frac{D^{m+1}f(0)(x_0^{m+1})}{(m+1)!} - \mu \frac{D^m f(0)(x_0^{m-1}, \frac{D^2 f(0)(x_0^2)}{2!})}{m!} \right\| \\
&= \left| \frac{D^m h(0)(x_0^m)}{m!} - \mu \frac{D^{m-1}h(0)(x_0^{m-1})}{(m-1)!} \frac{Dh(0)x_0}{1!} \right| \\
&= \left| T_x\left(\frac{D^{m+1}f(0)(x_0^{m+1})}{(m+1)!}\right) - \mu T_x\left(\frac{D^m f(0)(x_0^m)}{m!}\right) T_x\left(\frac{D^2 f(0)(x_0^2)}{2!}\right) \right| \\
&= |a_{m+1} - \mu a_m a_2| \\
&\le |1 - \mu|, \quad m = 2, \ldots, \nu,
\end{aligned}$$

即

$$\left\| \frac{D^{m+1}f(0)(x^{m+1})}{(m+1)!} - \mu \frac{D^m f(0)(x^{m-1}, \frac{D^2 f(0)(x^2)}{2!})}{m!} \right\| \le |1 - \mu| \|x\|^{m+1}, \quad x \in E, \ m = 2, \ldots, \nu.$$

最后由 (1.1) 知所证结论成立.

考虑映射

$$f(x)=\frac{x}{1-T_u(x)},\quad x\in E, \tag{3.4}$$

其中 u 固定, 且 $\|u\|=1$, 则不难验证 $f\in Q_{\mathbb{B}}(E)$, 且

$$\frac{D^mf(0)(x^m)}{m!}=x(T_u(x))^{m-1},\quad x\in E.$$

取 $x=ru\,(0\le r<1)$, 则易知对 $\mu\in(-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty), m=2,3,\ldots$, 有

$$\left\|\frac{D^{m+1}f(0)(x^{m+1})}{(m+1)!}-\mu\frac{D^mf(0)(x^{m-1},\frac{D^2f(0)(x^2)}{2!})}{m!}\right\|=|1-\mu|r^{m+1}.$$

上式表明定理 3.1 的估计是精确的.

在定理 3.1 中令 $\mu=0$, 则直接得到下列推论.

推论 3.2 设 $\nu=2,3,\ldots$, $f(x)=x+\sum_{m=2}^{\nu+1}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=\nu+2}^{\infty}\frac{D^mf(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$ ($Q_{\mathbb{A}}(E)$ 或 $Q(E)$), 其中 $h:E\to\mathbb{C}$ 是全纯函数, 且 $h(0)=1$, 则

$$\left\|\frac{D^{m+1}f(0)(x^{m+1})}{(m+1)!}\right\|\le\|x\|^{m+1},\quad x\in E,\ m=2,\ldots,\nu,$$

且上述估计是精确的.

在定理 3.1 中再令 $\nu=2$, 则立即得到如下推论.

推论 3.3 设 $f(x)=x+\sum_{m=2}^{3}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=4}^{\infty}\frac{D^mf(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$ ($Q_{\mathbb{A}}(E)$ 或 $Q(E)$), 其中 $h:E\to\mathbb{C}$ 是全纯函数, 且 $h(0)=1$, 则

$$\left\|\frac{D^3f(0)(x^3)}{3!}-\mu\frac{D^2f(0)(x,\frac{D^2f(0)(x^2)}{2!})}{2!}\right\|\le|1-\mu|\|x\|^3,\quad x\in E,$$

且上述估计是精确的.

作为定理 3.1 的应用, 则有下面

定理 3.4 设 $\nu=2,3,\ldots$, $f(x)=x+\sum_{m=2}^{\nu+1}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=\nu+2}^{\infty}\frac{D^mf(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$ ($Q_{\mathbb{A}}(E)$ 或 $Q(E)$), 其中 $h:E\to\mathbb{C}$ 是全纯函数, 且 $h(0)=1$, 则

$$\left|\frac{\|D^{m+1}f(0)(x^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}\right|\le\frac{m-1}{m+1}\|x\|^{m+1},\quad x\in E,\ m=2,\ldots,\nu,$$

且上述估计是精确的.

证明 对 $x\in E$ 与 $m=2,\ldots,\nu$,

$$\left|\frac{|D^{m+1}f(0)(x^{m+1})|}{(m+1)!}-\frac{2}{m+1}|x|\frac{|D^mf(0)(x^m)|}{m!}\right|\le\frac{m-1}{m+1}|x|^{m+1}$$

等价于对 $x\in E$ 与 $m=2,\ldots,\nu$,

$$\frac{|D^{m+1}f(0)(x^{m+1})|}{(m+1)!}-\frac{2}{m+1}|x|\frac{|D^mf(0)(x^m)|}{m!}\le\frac{m-1}{m+1}|x|^{m+1}$$

与

$$\frac{2}{m+1}|x|\frac{|D^mf(0)(x^m)|}{m!}-\frac{|D^{m+1}f(0)(x^{m+1})|}{(m+1)!}\le\frac{m-1}{m+1}|x|^{m+1}.$$

因此证明分下面两步进行.

第一步 在定理 3.1 中令 $\mu=\frac{2}{m+1}$, 则由推论 3.2 知

$$\begin{aligned}&\frac{\|D^{m+1}f(0)(x^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}\\&\le\left\|\frac{D^{m+1}f(0)(x^{m+1})}{(m+1)!}-\frac{2}{m+1}\frac{D^mf(0)(x^{m-1},\frac{D^2f(0)(x^2)}{2!})}{m!}\right\|\\&\quad+\frac{2}{m+1}\frac{\|D^mf(0)(x^{m-1},\frac{D^2f(0)(x^2)}{2!})\|}{m!}-\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}\\&\le\frac{m-1}{m+1}\|x\|^{m+1}+\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}-\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}\\&=\frac{m-1}{m+1}\|x\|^{m+1},\quad x\in E,\ m=2,\dots,\nu.\end{aligned}$$

第二步 显见, 对 $m\ge3$, 有

$$\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}-\frac{\|D^{m+1}f(0)(x^{m+1})\|}{(m+1)!}\le\frac{m-1}{m+1}\|x\|^{m+1}.$$

若 $m=2$, 则对 $\frac{\|D^2f(0)(x^2)\|}{2!}\le\frac{1}{2}\|x\|^2$, 有

$$\frac{2}{3}\|x\|\frac{\|D^2f(0)(x^2)\|}{2!}-\frac{\|D^3f(0)(x^3)\|}{3!}\le\frac{1}{3}\|x\|^3,$$

且对 $\frac{\|D^2f(0)(x^2)\|}{2!}\ge\frac{1}{2}\|x\|^2$, 由推论 3.3 ($\mu=\frac{4}{3}$ 情形) 知

$$\begin{aligned}\frac{2}{3}\|x\|\frac{\|D^2f(0)(x^2)\|}{2!}-\frac{\|D^3f(0)(x^3)\|}{3!}&=\frac{2}{3}\|x\|\frac{\|D^2f(0)(x^2)\|}{2!}-\frac{4}{3}\frac{\|D^2f(0)(x,\frac{D^2f(0)(x^2)}{2!})\|}{2!}\\&\quad+\frac{4}{3}\frac{\|D^2f(0)(x,\frac{D^2f(0)(x^2)}{2!})\|}{2!}-\frac{\|D^3f(0)(x^3)\|}{3!}\\&\le\left\|\frac{D^3f(0)(x^3)}{3!}-\frac{4}{3}\frac{D^2f(0)(x,\frac{D^2f(0)(x^2)}{2!})}{2!}\right\|\\&\le\frac{1}{3}\|x\|^3.\end{aligned}$$

从而所证结论成立.

取映射 f 如 (3.4) 定义. 则直接验证 $f\in Q_{\mathbb{B}}(E)$, 且

$$\frac{D^mf(0)(x^m)}{m!}=x(T_u(x))^{m-1},\quad x\in E.$$

取 $x=ru\,(0\le r<1)$. 显然有

$$\left|\frac{\|D^{m+1}f(0)(x^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|x\|\frac{\|D^mf(0)(x^m)\|}{m!}\right|=\frac{m-1}{m+1}r^{m+1},\quad m=2,3,\dots.$$

因此定理 3.2 是精确的.

在定理 3.4 中取 $\nu=2$, 则直接得到下面推论.

推论 3.5 设 $f(x)=x+\sum_{m=2}^{3}x\frac{D^{m-1}h(0)(x^{m-1})}{(m-1)!}+\sum_{m=4}^{\infty}\frac{D^mf(0)(x^m)}{m!}\in Q_{\mathbb{B}}(E)$ ($Q_{\mathbb{A}}(E)$ 或 $Q(E)$), 其中 $h:E\to\mathbb{C}$ 是全纯函数, 且 $h(0)=1$, 则

$$\left|\frac{\|D^3f(0)(x^3)\|}{3!}-\frac{2}{3}\|x\|\frac{\|D^2f(0)(x^2)\|}{2!}\right|\le\frac{1}{3}\|x\|^3,\quad x\in E,$$

且上述估计是精确的.

4 $\mathbb{C}^n$ 中单位多圆柱上准凸映射推广的 Fekete–Szegö 不等式及其应用

本节给出单位多圆柱上一类准凸映射子族推广的 Fekete–Szegö 不等式, 即定理 4.1 与定理 4.5 将单位圆盘经典的 Fekete–Szegö 不等式推广至高维单位多圆柱情形.

定理 4.1 设 $\nu = 2,3,\ldots,\mu \in (-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty)$, $m = 2,3,\ldots,\nu$, 且 $f(z) = z + \sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$). 若对 $z \in U^n, m = 2,\ldots,\nu$; $s = 2,\ldots,\nu - m + 2$, 有

$$z_j\frac{D^m f_j(0)(z^{m-1},\frac{D^s g(0)(z^s)}{s!})}{m!} = \frac{D^m f_j(0)(z^m)}{m!}\frac{D^s g_j(0)(z^s)}{s!}, \tag{4.1}$$

其中 $g(z) = (Df(z))^{-1}(Df(z)z + D^2f(z)(z^2))$, 且 j 满足 $|z_j| = \|z\| = \max_{1\le l\le n}\{|z_l|\}$, 则对 $z \in U^n$, $m = 2,3,\ldots,\nu$, 有

$$\left\|\frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!} - \mu\frac{D^m f(0)(z^{m-1},\frac{D^2 f(0)(z^2)}{2!})}{m!}\right\| \le |1-\mu|\|z\|^{m+1},$$

且上述估计是精确的.

证明 因 $f(z) = z + \sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$, 则可定义

$$g(z) = (Df(z))^{-1}(Df(z)z + D^2f(z)(z^2)).$$

易验证

$$\Re e\left(\frac{g_j(z)}{z_j}\right) \ge 0, \quad z \in U^n,$$

其中 j 满足 $|z_j| = \|z\| = \max_{1\le l\le n}\{|z_l|\}$. 又定义

$$p_j(\lambda) = \begin{cases} \dfrac{\|z\|}{z_j}\dfrac{g_j(\lambda z_0))}{\lambda}, & \lambda \in U\setminus\{0\},\\ 1, & \lambda = 0, \end{cases}$$

其中 $z \in U^n\backslash\{0\}$ 固定, $z_0 = \frac{z}{\|z\|}$. 则有 $\Re e(p_j(\lambda)) > 0$, $\lambda \in U$, 且

$$\begin{aligned} p_j(\lambda) &= 1 + \sum_{m=2}^{\infty} b_{j,m-1}\lambda^{m-1}\\ &= 1 + \frac{\|z\|}{z_j}\sum_{m=2}^{\nu}\frac{D^m g_j(0)(z_0^m)}{m!}\lambda^{m-1} + \frac{\|z\|}{z_j}\sum_{m=\nu+1}^{\infty}\frac{D^m g_j(0)(z_0^m)}{m!}\lambda^{m-1}. \end{aligned}$$

比较上述等式两边的系数, 得

$$b_{j,m-1} = \frac{\|z\|}{z_j}\frac{D^m g_j(0)(z_0^m)}{m!}, \quad m = 2,3,\ldots,\nu.$$

记

$$a_{j,m} = \frac{\|z\|}{z_j}\frac{D^m f_j(0)(z_0^m)}{m!}, \quad m = 2,3,\ldots,\nu+1.$$

由引理 2.2 ($E = U^n$ 情形) 与 (4.1), 有

$$(m+1)ma_{j,m+1} = \sum_{k=1}^{m} k a_{j,k}b_{j,m+1-k}, \quad m = 2,3,\ldots,\nu.$$

注意到

$$|b_{j,m-1}| = \left| \frac{\|z\|}{z_j} \frac{D^m g_j(0)(z_0^m)}{m!} \right| \le 2, \quad m = 2, 3, \dots \tag{4.2}$$

(参见文 [6, 定理2.1.5]). 因此由 (2.6) 推得

$$|a_{j,m}| \le 1, \quad m = 2, 3, \dots, \nu. \tag{4.3}$$

类似于定理 3.1 的证明, 由 (4.2)–(4.3), 对 $\mu \in (-\infty, \frac{2}{m+1}]$, 有

$$\left| \frac{2m}{m+1} a_{j,m} a_{j,2} - a_{j,m+1} \right| \le \frac{m-1}{m+1}, \quad m = 2, 3, \dots, \nu, \tag{4.4}$$

$$|a_{j,m+1} - \mu a_{j,m} a_{j,2}| \le 1 - \mu, \quad m = 2, 3, \dots, \nu,$$

与对 $\mu \in [\frac{2m}{m+1}, +\infty)$, 有

$$|a_{j,m+1} - \mu a_{j,m} a_{j,2}| \le \mu - 1, \quad m = 2, 3, \dots, \nu.$$

总之, 由 (4.4) 对 $\mu \in (-\infty, \frac{2}{m+1}] \cup [\frac{2m}{m+1}, +\infty)$, $m = 2, 3, \dots, \nu$,

$$\left| \frac{D^{m+1} f_j(0)(z_0^{m+1})}{(m+1)!} - \mu \frac{D^m f_j(0)(z_0^{m-1}, \frac{D^2 f(0)(z_0^2)}{2!})}{m!} \right| \le |\mu - 1|$$

成立.

特别, 对 $z_0 \in (\partial U)^n$, 有

$$\left| \frac{D^{m+1} f_l(0)(z_0^{m+1})}{(m+1)!} - \mu \frac{D^m f_l(0)(z_0^{m-1}, \frac{D^2 f(0)(z_0^2)}{2!})}{m!} \right| \le |\mu - 1|, \quad l = 1, 2, \dots, n.$$

因每个

$$\frac{D^{m+1} f_l(0)(z^{m+1})}{(m+1)!} - \mu \frac{D^m f_l(0)(z^{m-1}, \frac{D^2 f(0)(z^2)}{2!})}{m!}$$

在 $\overline{U^n}$ 上全纯, 故由单位多圆柱上的最大模原理知

$$\left| \frac{D^{m+1} f_l(0)(z_0^{m+1})}{(m+1)!} - \mu \frac{D^m f_l(0)(z_0^{m-1}, \frac{D^2 f(0)(z_0^2)}{2!})}{m!} \right| \le |\mu - 1|, \quad z_0 \in \partial U^n, \ l = 1, 2, \dots, n.$$

因此推得

$$\left| \frac{D^{m+1} f_l(0)(z^{m+1})}{(m+1)!} - \mu \frac{D^m f_l(0)(z^{m-1}, \frac{D^2 f(0)(z^2)}{2!})}{m!} \right| \le |\mu - 1| \|z\|^{m+1}, \quad z \in U^n, \ l = 1, 2, \dots, n.$$

从而, 对 $z \in U^n, m = 2, 3, \dots, \nu$, 有

$$\left\| \frac{D^{m+1} f(0)(z^{m+1})}{(m+1)!} - \mu \frac{D^m f(0)(z^{m-1}, \frac{D^2 f(0)(z^2)}{2!})}{m!} \right\| \le |m + 1 - 2m\mu| \|z\|^{m+1}.$$

考虑映射

$$f(z) = \left(\frac{z_1}{1 - z_1}, \frac{z_2}{1 - z_2}, \dots, \frac{z_n}{1 - z_n} \right)', \quad z \in U^n.$$

易验证 $f \in Q_{\mathbb{B}}(U^n)$. 取 $z = (r, r, \dots, r)$ $(0 \le r < 1)$. 则对 $\mu \in (-\infty, \frac{2}{m+1}] \cup [\frac{2m}{m+1}, +\infty)$, $m = 2, 3, \dots$, 有

$$\left\| \frac{D^{m+1} f(0)(z^{m+1})}{(m+1)!} - \mu \frac{D^m f(0)(z^{m-1}, \frac{D^2 f(0)(z^2)}{2!})}{m!} \right\| = |\mu - 1| r^{m+1}.$$

因此定理 4.1 的估计式是精确的.

不难验证两个推论的附加条件满足定理 4.1 的假设 (略去证明).

推论 4.2 设 $\nu = 2,3,\ldots,\mu \in (-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty)$, $m = 2,3,\ldots,\nu$, $f(z) = z + \sum_{m=2}^{\nu} z\frac{D^{m-1}h(0)(z^{m-1})}{(m-1)!} + \sum_{m=\nu+1}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 其中 $h: U^n \to \mathbb{C}$ 是全纯函数, 且 $h(0) = 1$, 则对 $z \in U^n$, $m = 2,3,\ldots,\nu$, 有

$$\left\| \frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!} - \mu\frac{D^m f(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!} \right\| \le |1-\mu|\|z\|^{m+1},$$

且上述估计是精确的.

推论 4.3 设 $\nu = 2,3,\ldots,\mu \in (-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty)$, $m = 2,3,\ldots,\nu$. 若 $f(z) = z + \sum_{m=2}^{\nu}(\frac{f_1^{(m)}(0)}{m!}z_1^m,\ldots,\frac{f_n^{(m)}(0)}{m!}z_n^m)' + \sum_{m=\nu+1}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), $\nu = 2,3,\ldots$, 则对 $z \in U^n$, $m = 2,3,\ldots,\nu$, 有

$$\left\| \frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!} - \mu\frac{D^m f(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!} \right\| \le |1-\mu|\|z\|^{m+1},$$

且上述估计是精确的.

再在定理 4.1 中取 $\nu = 2$, 则直接得

推论 4.4 设 $\mu \in (-\infty,\frac{2}{3}]\cup[\frac{4}{3},+\infty), f(z) = z + \sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$). 若

$$z_j\frac{D^2 f_j(0)(z,\frac{D^2f(0)(z^2)}{2!})}{2!} = \frac{D^2 f_j(0)(z^2)}{2!}\frac{D^2 f_j(0)(z^2)}{2!}, \quad z \in U^n,$$

其中 j 满足 $|z_j| = \|z\| = \max_{1\le l\le n}\{|z_l|\}$, 则

$$\left\| \frac{D^3 f(0)(z^3)}{3!} - \mu\frac{D^2 f(0)(z,\frac{D^2f(0)(z^2)}{2!})}{2!} \right\| \le |1-\mu|\|z\|^3, \quad z \in U^n,$$

且上述估计是精确的.

仿照定理 3.4 和定理 4.1 的证明, 则有如下单位多圆柱形式的推广的 Fekete–Szegö 不等式 (略去证明).

定理 4.5 设 $\nu = 2,3,\ldots,\mu \in (-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty)$, $m = 2,3,\ldots,\nu$, 且 $f(z) = z + \sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$). 若对 $z \in U^n, m = 2,\ldots,\nu$; $s = 2,\ldots,\nu - m + 2$, 有

$$z_j\frac{D^m f_j(0)(z^{m-1},\frac{D^s g(0)(z^s)}{s!})}{m!} = \frac{D^m f_j(0)(z^m)}{m!}\frac{D^s g_j(0)(z^s)}{s!},$$

其中 j 满足 $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$, 且 $g(z) = (Df(z))^{-1}(Df(z)z + D^2f(z)(z^2))$, 则

$$\left| \frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!} - \frac{2}{m+1}\|z\|\frac{\|D^m f(0)(z^m)\|}{m!} \right| \le \frac{m-1}{m+1}\|z\|^{m+1}, \quad z \in U^n,\ m = 2,\ldots,\nu,$$

且上述估计是精确的.

同样不难验证两个推论的附加条件满足定理定理 4.5 的假设 (略去证明).

推论 4.6 设 $\nu = 2,3,\ldots$, $f(z) = z + \sum_{m=2}^{\nu} z\frac{D^{m-1}h(0)(z^{m-1})}{(m-1)!} + \sum_{m=\nu+1}^{\infty}\frac{D^m f(0)(z^m)}{m!} \in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 其中 $h: U^n \to \mathbb{C}$ 是全纯函数, 且 $h(0) = 1$, 则对 $z \in U^n, m =$

$2,3,\ldots,\nu$, 有

$$\left|\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|z\|\frac{\|D^mf(0)(z^m)\|}{m!}\right|\le\frac{m-1}{m+1}\|z\|^{m+1},$$

且上述估计是精确的.

推论 4.7 设 $\nu=2,3,\ldots$. 若

$$f(z)=z+\sum_{m=2}^{\nu}\left(\frac{f_1^{(m)}(0)}{m!}z_1^m,\ldots,\frac{f_n^{(m)}(0)}{m!}z_n^m\right)'+\sum_{m=\nu+1}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_B(U^n)\ (Q_A(U^n)\text{或}Q(U^n)),$$

则对 $z\in U^n,\ m=2,3,\ldots,\nu$, 有

$$\left|\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|z\|\frac{\|D^mf(0)(z^m)\|}{m!}\right|\le\frac{m-1}{m+1}\|z\|^{m+1},$$

且上述估计是精确的.

在定理 4.5 中取 $\nu=2$, 即得

推论 4.8 设 $\mu\in(-\infty,\frac{2}{3}]\cup[\frac{4}{3},+\infty)$, 且 $f(z)=z+\sum_{m=2}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ $(Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n))$. 若对 $z\in U^n$, 有

$$z_j\frac{D^2f_j(0)(z,\frac{D^2f(0)(z^2)}{2!})}{2!}=\frac{D^2f_j(0)(z^2)}{2!}\frac{D^2f_j(0)(z^2)}{2!},$$

其中 j 满足 $|z_j|=\|z\|=\max_{1\le l\le n}\{|z_l|\}$, 则

$$\left|\frac{\|D^3f(0)(z^3)\|}{3!}-\frac{2}{3}\|z\|\frac{\|D^2f(0)(z^2)\|}{2!}\right|\le\frac{1}{3}\|z\|^3,\quad z\in U^n,$$

且上述估计是精确的.

现讨论定理 4.1 更为一般的情形.

定理 4.9 设 $\mu\in(-\infty,\frac{2}{m+1}]$, $f(z)=z+\sum_{m=2}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ $(Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n))$, 且 $\sup_{\|z\|=1,\|w\|=1}\frac{\|D^mf(0)(z^{m-1},w)\|}{m!}=\sup_{\|z\|=1}\frac{\|D^mf(0)(z^m)\|}{m!}$, $m=2,3,\ldots$, 则对 $z\in U^n,m=2,3,\ldots$, 有

$$\left\|\frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!}-\mu\frac{D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!}\right\|\le(1-\mu)\|z\|^{m+1},$$

且上述估计是精确的.

证明 由定理 4.9 的假设知, 对 $z\ne0$, $\frac{D^sg(0)(z^s)}{s!}\ne0$, 有

$$\frac{\|D^mf(0)(z_0^{m-1},w_0)\|}{m!}\le\sup_{\|z_0\|=1}\frac{\|D^mf(0)(z_0^m)\|}{m!}\le1,$$

其中 $z_0=\frac{z}{\|z\|},w_0=\frac{\frac{D^sg(0)(z^s)}{s!}}{\frac{\|D^sg(0)(z^s)\|}{s!}}$. 注意到

$$\frac{\|D^mg(0)(z^m)\|}{m!}\le2\|z\|^m,\quad m=2,3,\ldots$$

(参见文 [11, 引理2]). 于是对 $z\in U^n,m,s=2,3,\ldots$, 有

$$\frac{\|D^mf(0)(z^{m-1},\frac{D^sg(0)(z^s)}{s!})\|}{m!}\le\|z\|^{m-1}\frac{\|D^sg(0)(z^s)\|}{s!}\le2\|z\|^{m+s-1}.\tag{4.5}$$

另一方面, 由引理 2.1 ($E=U^n$ 情形) 与 (4.5), 对 $\mu\in(-\infty,\frac{2}{m+1}]$, $m=2,3,\ldots$, 有

$$\begin{aligned}
&\left\|\frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!}-\mu\frac{D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!}\right\|\\
&\quad\le\left\|\frac{1}{(m+1)m}\left(\frac{D^{m+1}g(0)(z^{m+1})}{(m+1)!}+2\frac{D^2f(0)(z,\frac{D^mg(0)(z^m)}{m!})}{2!}+\cdots\right.\right.\\
&\qquad\left.\left.+(m-1)\frac{D^{m-1}f(0)(z^{m-2},\frac{D^mg(0)(z^3)}{3!})}{(m-1)!}\right)+\left(\frac{2}{m+1}-\mu\right)\frac{D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!}\right\|\\
&\quad\le\frac{1}{(m+1)m}(2\|z\|^{m+1}+2\cdot2\|z\|^{m+1}+\cdots+2\cdot(m-1)\|z\|^{m+1})+\left(\frac{2}{m+1}-\mu\right)\|z\|^{m+1}\\
&\quad=\frac{m-1}{m+1}\|z\|^{m+1}+\left(\frac{2}{m+1}-\mu\right)\|z\|^{m+1}\\
&\quad=(1-\mu)\|z\|^{m+1}.
\end{aligned}$$

于是所证结论成立. 定理 4.9 精确性的讨论类似于定理 4.1.

注 4.10 引理 2.4 表明定理 4.9 的限制条件较弱.

作为定理 4.9 的应用, 有如下定理.

定理 4.11 设 $f(z)=z+\sum_{m=2}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 且

$$\sup_{\|z\|=1,\|w\|=1}\frac{\|D^mf(0)(z^{m-1},w)\|}{m!}=\sup_{\|z\|=1}\frac{\|D^mf(0)(z^m)\|}{m!},\quad m=2,3,\ldots,$$

则

$$\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}\le\frac{m-1}{m+1}\|z\|^{m+1},\quad m=2,3,\ldots,\tag{4.6}$$

与

$$\frac{2}{m+1}\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}-\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}\le\frac{m-1}{m+1}\|z\|^{m+1},\quad m=3,4,\ldots,\tag{4.7}$$

且上述估计是精确的.

证明 注意到对 $z\ne0$, $\frac{D^2f(0)(z^2)}{2!}\ne0$, 由定理 4.9 ($\mu=0$ 情形), 有

$$\frac{\|D^mf(0)(z_0^{m-1},w_0)\|}{m!}\le\sup_{\|z_0\|=1}\frac{\|D^mf(0)(z_0^m)\|}{m!}\le1,$$

其中 $z_0=\frac{z}{\|z\|}$, $w_0=\frac{\frac{D^sf(0)(z^2)}{2!}}{\frac{\|D^2f(0)(z^2)\|}{2!}}$, 对 $z\in U^n,m=2,3,\ldots$, 于是

$$\frac{\|D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})\|}{2!}\le\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}.\tag{4.8}$$

在定理 4.9 中取 $\mu=\frac{2}{m+1}$, 由 (4.8), 得

$$\begin{aligned}
&\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|z\|\frac{\|D^2f(0)(z^2)\|}{2!}\\
&\quad\le\left\|\frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!}-\frac{2}{m+1}\frac{D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!}\right\|\\
&\qquad+\frac{2}{m+1}\frac{\|D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})\|}{m!}-\frac{2}{m+1}\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}
\end{aligned}$$

$$\leq \frac{m-1}{m+1}\|z\|^{m+1}+\frac{2}{m+1}\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}-\frac{2}{m+1}\|z\|^{m-1}\frac{\|D^2f(0)(z^2)\|}{2!}$$
$$=\frac{m-1}{m+1}\|z\|^{m+1},$$

故 (4.6) 成立. 此外, (4.7) 显然成立. 定理 4.11 精确性的讨论类似于定理 4.1.

特别地, 在定理 4.11 中取 $m=2$, 直接得

推论 4.12 设 $f\in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 且

$$\sup_{\|z\|=1,\|w\|=1}\frac{\|D^2f(0)(z,w)\|}{2!}=\sup_{\|z\|=1}\frac{\|D^2f(0)(z^2)\|}{2!},$$

则

$$\frac{\|D^3f(0)(z^3)\|}{3!}-\frac{2}{3}\|z\|\frac{\|D^2f(0)(z^2)\|}{2!}\leq\frac{1}{3}\|z\|^3,$$

且上述估计是精确的.

由定理 4.1、定理 4.5、定理 4.9 与定理 4.11, 很自然地提出如下两个公开问题.

问题 4.13 设 $\mu\in(-\infty,\frac{2}{m+1}]\cup[\frac{2m}{m+1},+\infty), m=2,3,\ldots, f(z)=z+\sum_{m=2}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 则对 $z\in U^n,\ m=2,3,\ldots$, 有

$$\left\|\frac{D^{m+1}f(0)(z^{m+1})}{(m+1)!}-\mu\frac{D^mf(0)(z^{m-1},\frac{D^2f(0)(z^2)}{2!})}{m!}\right\|\leq|1-\mu|\|z\|^{m+1},$$

且上述估计是精确的.

问题 4.14 设 $f(z)=z+\sum_{m=2}^{\infty}\frac{D^mf(0)(z^m)}{m!}\in Q_{\mathbb{B}}(U^n)$ ($Q_{\mathbb{A}}(U^n)$ 或 $Q(U^n)$), 则

$$\left|\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}-\frac{2}{m+1}\|z\|\frac{\|D^mf(0)(z^m)\|}{m!}\right|\leq\frac{m-1}{m+1}\|z\|^{m+1},\quad z\in U^n,m=2,3,\ldots,$$

且上述估计是精确的.

致谢 本文作者感谢评审人中肯的建议.

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ACTA MATHEMATICA SINICA, CHINESE SERIES
No. 5 Vol. 68, Sep. 2025

CONTENTS

Existence of Generalized Kazdan–Warner Equation with Non-negative Prescribed Function ..... Guoyi Yang & Xiaobao Zhu (745)

Congruences for Restricted 4-Colored Partitions ..... Wu-Xia Ma & Yong-Gao Chen (755)

Generalized Derivations of δ-BiHom-Jordan Lie Supertriple Systems ..... Shuo Song & Liming Tang (765)

Existence and Uniqueness of Strong Solutions for a Class of Mean-field Backward Stochastic Differential Equations with Non-Lipschitz Coefficients ..... Tao Hao (781)

Construction of Confidence Intervals for Relative Risk in Matched-Pair Design ..... Huan He, Xiao He, Liping Zhang & Maozai Tian (799)

The Iterative Roots of a Flat-Top Anti-Bimodal Continuous Self-Map on an Interval ..... Taixiang Sun, Bin Qin & Caihong Han (820)

Spectral Characterizations of F-Willmore Hypersurfaces ..... Dengyun Yang, Jinguo Zhang & Yongqian Tao (831)

An Enumeration Problem for Permutation Polynomials ..... Juan Li, Huanhuan Guan & Danyao Wu (839)

The Einstein–Hilbert Action of a Class of Dirac Operators with Torsion ..... Jian Wang, Tong Wu & Kaihua Bao (847)

On the Number of Rational Points of Triangular Algebraic Varieties over Finite Fields ..... Siao Hong & Guangyan Zhu (857)

The Generalized Fekete and Szegö Inequality for Quasi-convex Mappings in Several Complex Variables and Its Applications ..... Xiaosong Liu & Haichou Li (872)

Abstracts of Acta Mathematica Sinica, English Series, Nos. 7 & 8, Vol. 41 (2025) ..... (i–viii)

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数 学 学 报 中文版

2025 年 9 月 第 68 卷 第 5 期

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http://www.ActaMath.com

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编辑出版 《数 学 学 报》 编 辑 委 员 会

印刷装订 北 京 科 信 印 刷 有 限 公 司

国内发行 北 京 报 刊 发 行 局 或 编 辑 部

国外发行 中 国 国 际 图 书 贸 易 总 公 司

(中国国际书店) 北京 399 信箱

ISSN 0583-1431

邮发代号: 2-502 国外刊号: BM 48 定价: 150.00 元

广告许可证: 京海工商广字第 0360 号 公 开 发 行

Adv. Appl. Clifford Algebras (2025) 35:35
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2025
0188-7009/030001-41
published online June 25, 2025
https://doi.org/10.1007/s00006-025-01395-6

Advances in Applied Clifford Algebras

Boundedness of Multiparameter Forelli–Rudin Type Operators on Product L^p Spaces over Tubular Domains

Lvchang Li*, Yuheng Liang and Haichou Li*

Communicated by Irene Sabadini

Abstract. In this paper, we introduce and study two classes of multiparameter Forelli–Rudin type operators from $L^{\vec{p}}(\mathcal{D})$ to $L^{\vec{q}}(\mathcal{D})$, especially on their boundedness, where $L^{\vec{p}}(\mathcal{D})$ and $L^{\vec{q}}(\mathcal{D})$ are both weighted Lebesgue spaces over the Cartesian product of two tubular domains T_B, with mixed-norm and appropriate weights. We completely characterize the boundedness of these two operators when $1 \leq \vec{p} \leq \vec{q} < \infty$. Moreover, we provide the necessary and sufficient condition of the case that $\vec{q} = (\infty, \infty)$. As an application, we obtain the boundedness of three common classes of integral operators, including the weighted multiparameter Bergman-type projection and the weighted multiparameter Berezin-type transform.

Mathematics Subject Classification. 47A05, 47A07, 47A13, 47A30, 47B38, 47B01, 47B91.

Keywords. Forelli–Rudin type operators, Mixed-norm Lebesgue space, Tubular domain, Boundedness.

Contents

1. Introduction 2
2. Preliminaries 4
3. Basic Lemmas 5

This article is part of the Joint topical Collection of Advances in Applied Clifford Algebras and Complex Analysis and Operator Theory in honor of Frank Sommen, edited by Paula Cerejeiras, Hendrik de Bie, Tao Qian, Irene Sabadini, Danielle Struppa, and Adrian Vajiac.

This work was supported by the NSF of Guangdong Province, China (Grant No. 2025A1515011213) and NNSF of China (Grant No. 12326407).

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4. The Necessity for the Boundedness of $T_{\vec{a},\,\vec{b},\,\vec{c}}$ 9
5. The Sufficiency for the Boundedness of $S_{\vec{a},\,\vec{b},\,\vec{c}}$ 20
6. The Proof of Main Theorems 32
7. Applications 35
References 39

1. Introduction

The study of the analytic properties of integral operators has always been a very active problem. In recent years, many scholars have studied its boundedness, compactness and other analytical properties, and obtained a series of important results, which can be referred to [5]. The Forelli–Rudin operator was first introduced by Forelli and Rudin [7] in 1974 as a special class of integral operators. They play an important role in the theory of function spaces, especially in the study of Bergman spaces and Hardy spaces. Initially, this research mainly focused on the operator theory of Bergman space and Hardy space on the unit disk and the unit ball. Subsequently, it is gradually extended to a wider range of function spaces, such as weighted Bergman space and Sobolev space. Researchers have found that the behavior of the Forelli–Rudin operator in these spaces can help to understand more general function space theory.

Kurens and Zhu [14] introduced the Forelli–Rudin type integral operators in 2006, which are defined as follows. For any given $a, b, c \in \mathbb{R}$,

$$T_{a,\,b,\,c}\, f(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|u|^2)^b}{(1-\langle z,u\rangle)^c} f(u)\, dV(u)$$

and

$$S_{a,\,b,\,c}\, f(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|u|^2)^b}{|1-\langle z,u\rangle|^c} f(u)\, dV(u).$$

The above operators are defined on the unit ball. Stein [6] first proved that the operator $T_{0,\,0,\,n+1}$ is bounded on $L^p(\mathbb{B}_n)$, where $1 < p < \infty$. Forelli–Rudin [7] proved that $T_{0,\sigma+it,n+1+\sigma+it}$ is bounded on $L^p(\mathbb{B}_n)$ if and only if $(\sigma+1)p > 1$, where $1 < p < \infty, \sigma > -1, t \in \mathbb{R}$, i is an imaginary unit. Zhu [30, Theorem 3.11] gave the necessary and sufficient conditions for $T_{a,\,b,\,c}$ and $S_{a,\,b,\,c}$ to be bounded on $L^p(\mathbb{D}, dA_\alpha)$, where $1 < p < \infty$, $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dV(z), \alpha > -1$. Later, this conclusion was extended to the high-dimensional case by Kures and Zhu [14]. For more results in this direction, please refer to [22,23,28,29]. Kaptanoğlu and Üreyen [15] give a complete characterization of the boundedness of the standard weighted integral operators induced by the Bergman-Besov kernels under the unit ball. In addition, it also includes the study of Forelli–Rudin operators on special domains (such as unbounded domains), revealing the behavior of operators in different geometric backgrounds. For example, Cheng et al.[2] gave the boundedness of $T_{0,\,0,\,c}$ from L^p to L^q in the upper half plane when $(p,q) \in [1,\infty] \times [1,\infty]$. In [18], Liu et al. generalized the results of Cheng et al. to the Siegel upper

half space, and discussed the L^p-L^q boundedness of the operator $T_{0,\,0,\,c}$. In [27], Zhou et al. gave the necessary and sufficient conditions for the L^p_α-L^q_β boundedness of Forelli–Rudin type integral operators on Siegel upper half space, and gave the connection between the unit ball and the boundedness of Forelli–Rudin type integral operators on Siegel upper half space. For more results in the Siegel upper half space, please refer to [17,26].

The mixed norm Lebesgue space $L^{\vec{p}}$ as a natural generalization of the classical Lebesgue space L^p was first introduced by Benedek [1] in 1961. Its definition and properties combine multiple Lebesgue integrals in different directions, which can better describe the behavior of multidimensional functions. Mixed norm Lebesgue spaces are increasingly showing their importance in many analysis and application fields, especially in multivariate function analysis and partial differential equation research. By combining Lebesgue integrals in different directions, the mixed norm Lebesgue space can better describe and deal with the behaviors of complex functions. After that, many scholars have studied it. The research on this aspect can refer to the relevant literature [25]. There is also progress in the study of Forelli–Rudin type operators on Lebesgue spaces with mixed norm. Recently, Huang et al.[10] studied the boundedness of multi-parameter Forelli–Rudin type operators on weighted Lebesgue spaces $L^{\vec{p}}_{\vec{\alpha}}(\mathbb{B}_n \times \mathbb{B}_n)$ with mixed norm on the Cartesian product of two unit balls, and gave a series of conclusions. The object of this paper is the boundedness of multi-parameter Forelli–Rudin type operators on the weighted Lebesgue space $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ with mixed norm on the Cartesian product of two tubular domains. It also includes the boundedness of multi-parameter Forelli–Rudin type operators on $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ when $\vec{q} := (\infty, \infty)$. The detailed conclusions are shown in Sect. 6 of this paper.

In addition, multiparameter theory is also an important branch of mathematics. It provides a basis for dealing with complex mathematical problems involving multiple variables or parameters, and plays an important role in harmonic analysis, complex analysis, partial differential equations and other fields. For example, the singular integral operator studied by many scholars in harmonic analysis, the product singular integral studied by Fefferman et al. [8,13]. Specifically, in 1982, Fefferman and Stein [6] extended the convolution Calderón–Zygmund operator to the two-parameter case and obtained the boundedness on the product Lebesgue space. Gundy and Stein [9] first introduced the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Subsequently, Chang and Fefferman [3,4] established the Calderón–Zygmund decomposition theory of the product space and derived a series of conclusions. For more applications of multi-parameters in the above fields, see reference [11,20,21,24].

The rest of this article is organized as follows. In the Sect. 1, we will explain the basic concepts and symbolic terms. In the Sect. 3, we give some basic lemmas, including Schur test, which is an important tool to verify the boundedness of integral operators and will run through the full text. In the Sect. 4, we will give the necessary conditions for the boundedness of the operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$, and get a series of conclusions, see lemma 4.1 to lemma 4.13. In the Sect. 5, we give the sufficient conditions for the boundedness of the

operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$, see Lemma 5.1 to Lemma 5.13. Since the boundedness of the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ will inevitably lead to the boundedness of the operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$, in the Sect. 6, we link the above lemmas and give our main Theorems 6.1–6.13 in this paper. In the last section, we apply the obtained theorem to three kinds of integral operators, including the famous Bergman-type projection and Berezin-type transformation, and obtain the boundedness of three kinds of integral operators.

2. Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For any two points $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$ in $\mathbb{C}^n$, we write

$$z\cdot\bar{w}:=z_1\bar{w}_1+\cdots+z_n\bar{w}_n,$$

$$z'^2=z'\cdot z'=z_1^2+z_2^2+\cdots+z_n^2$$

and

$$|z|:=\sqrt{z\cdot\bar{z}}=\sqrt{|z_1|^2+\cdots+|z_n|^2}.$$

The tubular domain T_B of $\mathbb{C}^n$ with base B, is defined as follows:

$$T_B=\left\{z=x+iy\in\mathbb{C}^n|x\in\mathbb{R}^n,\ y\in B\subseteq\mathbb{R}^n\right\},$$

where

$$B=\left\{(y',y_n)=(y_1,\ldots,y_{n-1},y_n)\in\mathbb{R}^n\left|y'^2:=y_1^2+\cdots+y_{n-1}^2<y_n\right.\right\}.$$

For the convenience of writing, we use $\mathcal{D}$ to denote $T_B\times T_B$.

For any given $\vec{p}:=(p_1,p_2)\in(0,\infty)\times(0,\infty)$ and $\vec{\alpha}:=(\alpha_1,\alpha_2)\in(-1,\infty)\times(-1,\infty)$, we define the Lebesgue space $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ with weighted mixed-norm on $T_B\times T_B$. It consists of all Lebesgue measurable functions f on $\mathcal{D}$ such that the norm

$$\|f\|_{\vec{p},\vec{\alpha}}=\left\{\int_{T_B}\left(\int_{T_B}|f(z,w)|^{p_1}\,dV_{\alpha_1}(z)\right)^{\frac{p_2}{p_1}}dV_{\alpha_2}(w)\right\}^{\frac{1}{p_2}}$$

is finite, where for any $i\in\{1,2\}$, $dV_{\alpha_i}(z)=\rho(z)^{\alpha_i}dV(z)$, with $\rho(z):=y_n-|y'|^2$ and dV denoting the Lebesgue volume measure on $\mathbb{C}^n$.

In particular, when $p_1=p_2=p$ and $\alpha_1=\alpha_2=\alpha$, the space $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ goes back to the weighted Lebesgue space $L^p_\alpha(\mathcal{D})$.

Similarly, we define the space of all essentially bounded functions on $\mathcal{D}$, denoted as $L^\infty(\mathcal{D})$.

Same as the well-known classical L^p space, the weighted mixed-norm Lebesgue space $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ that we define is a Banach space under the norm $\|\cdot\|_{\vec{p},\vec{\alpha}}$ when $\vec{p}=(p_1,p_2)\in[1,\infty)\times[1,\infty)$.

Next, we introduce two important integral operators $T_{\vec{a},\,\vec{b},\,\vec{c}}$ and $S_{\vec{a},\,\vec{b},\,\vec{c}}$ studied in this paper.

For any $\vec{a} := (a_1, a_2), \vec{b} := (b_1, b_2), \vec{c} := (c_1, c_2) \in \mathbb{R}^2$, two classes of integral operators are defined by

$$T_{\vec{a},\,\vec{b},\,\vec{c}}\, f(z,w) = \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(u)^{b_1}\rho(\eta)^{b_2}}{\rho(z,u)^{c_1}\rho(w,\eta)^{c_2}} f(u,\eta)\,dV(u)dV(\eta)$$

and

$$S_{\vec{a},\,\vec{b},\,\vec{c}}\, f(z,w) = \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(u)^{b_1}\rho(\eta)^{b_2}}{|\rho(z,u)|^{c_1}|\rho(w,\eta)|^{c_2}} f(u,\eta)\,dV(u)dV(\eta),$$

where

$$\rho(z,u) := \frac{1}{4}\left(\left(z' - \overline{u'}\right)^2 - 2i\left(z_n - \overline{u_n}\right)\right).$$

In this paper, we stipulate that the mixed norm Lebesgue space $L^{\vec{p}}$ is denoted by L^∞ when only two indexes are infinite, and the rest are denoted by $L^{\vec{p}}$.

For the convenience of writing, we write $p_- := \min\{p_1, p_2\}$, $p_+ := max\{p_1, p_2\}$, $q_- := min\{q_1, q_2\}$ and $q_+ := max\{q_1, q_2\}$.

Throughout the paper we use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. The notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$, and the notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

3. Basic Lemmas

In this section, we will introduce several key lemmas, which will play an important role in the proof of the main theorems.

The following Lemmas 3.1 and 3.2 are derived from [19] and [16] respectively, which play an important role in integral estimation.

Lemma 3.1. *Let $r, s > 0, t > -1$ and $r + s - t > n + 1$, then*

$$\int_{T_B}\frac{\rho(w)^t}{\rho(z,w)^r\rho(w,u)^s}dV(w) = \frac{C_1(n,r,s,t)}{\rho(z,u)^{r+s-t-n-1}}$$

for all $z, u \in T_B$, where

$$C_1(n,r,s,t) = \frac{2^{n+1}\pi^n\Gamma(1+t)\Gamma(r+s-t-n-1)}{\Gamma(r)\Gamma(s)}.$$

In particular, let $s, t \in \mathbb{R}$, if $t > -1, s - t > n + 1$, then

$$\int_{T_B}\frac{\rho(w)^t}{|\rho(z,w)|^s}dV(w) = \frac{C_1(n,s,t)}{\rho(z)^{s-t-n-1}}.$$

Otherwise, the above equation is infinity.

Lemma 3.2. *For any $z, w \in T_B$, we have*

$$2|\rho(z,w)| \ge \max\{\rho(z), \rho(w)\}.$$

The following Lemma 3.3 is an important result in real analysis. Detailed proof can refer to [12].

Lemma 3.3. *Suppose $(X, d\mu)$ is a σ-finite measure space, $1 \leq p < \infty$, and $1/p + 1/q = 1$. Let G be a complex-valued function defined on $X \times X$ and T be the integral operator defined by*

$$Tf(x) = \int_X G(x,y) f(y) \, d\mu(y).$$

If the operator T is bounded on $L^p(X, d\mu)$, then its adjoint T^* is the integral oparator

$$T^* f(x) = \int_X \overline{G(y,x)} f(y) \, d\mu(y)$$

on $L^q(X, d\mu)$.

The following lemma 3.4 is a further conclusion on the basis of lemma 3.3.

Lemma 3.4. *Let $\vec{p} := (p_1, p_2) \in [1,\infty] \times [1,\infty]$ and $\vec{q} := (q_1, q_2) \in [1,\infty] \times [1,\infty]$. If the integral operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, then its adjoint operator $T^*_{\vec{a},\,\vec{b},\,\vec{c}}$ defined by setting*

$$T^*_{\vec{a},\,\vec{b},\,\vec{c}} f(z,w) = \rho(z)^{b_1-\alpha_1} \rho(w)^{b_2-\alpha_2} \int_{T_B}\int_{T_B} \frac{\rho(u)^{\beta_1+a_1} \rho(\eta)^{\beta_2+a_2}}{\rho(z,u)^{c_1} \rho(w,\eta)^{c_2}} f(u,\eta) \, dV(u) dV(\eta)$$

is bounded from $L^{\vec{q}'}_{\vec{\beta}}(\mathcal{D})$ to $L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$.

Proof. It is easy to get the definition of operator $T^*_{\vec{a},\,\vec{b},\,\vec{c}}$ from Lemma 3.3.

Let $g(z,w) \in L^{\vec{q}'}_{\vec{\alpha}}(\mathcal{D})$, it follows from the Fubini theorem and the Hölder inequality of the mixed norm that

$$\begin{aligned}
&\|T^*_{\vec{a},\,\vec{b},\,\vec{c}} g(z,w)\|_{\vec{p}',\vec{\alpha}} \\
&= \sup_{\|f\|_{\vec{p},\vec{\alpha}}=1} \left| \int_{T_B}\int_{T_B} T^*_{\vec{a},\,\vec{b},\,\vec{c}} g(z,w) \overline{f(z,w)} dV_{\alpha_1}(z) dV_{\alpha_2}(w) \right| \\
&= \sup_{\|f\|_{\vec{p},\vec{\alpha}}=1} \left| \int_{T_B}\int_{T_B}\int_{T_B}\int_{T_B} \frac{\rho(u)^{\beta_1+a_1} \rho(\eta)^{\beta_2+a_2}}{\rho(z,u)^{c_1} \rho(w,\eta)^{c_2}} \right. \\
&\quad \left. \times g(u,\eta) \, dV(u) \, dV(\eta) \overline{f(z,w)} dV_{\alpha_1}(z) \, dV_{\alpha_2}(w) \right| \\
&= \sup_{\|f\|_{\vec{p},\vec{\alpha}}=1} \left| \int_{T_B}\int_{T_B} g(u,\eta) \overline{T_{\vec{a},\,\vec{b},\,\vec{c}} f(u,\eta)} dV_{\beta_1}(u) dV_{\beta_2}(\eta) \right| \\
&\leq \sup_{\|f\|_{\vec{p},\vec{\alpha}}=1} \|T_{\vec{a},\,\vec{b},\,\vec{c}} f\|_{\vec{q},\vec{\beta}} \, \|g\|_{\vec{q}',\vec{\beta}} \\
&\lesssim \|g\|_{\vec{q}',\vec{\beta}}.
\end{aligned}$$

Therefore, the operator $T^*_{\vec{a},\,\vec{b},\,\vec{c}}$ is a bounded operator from $L^{\vec{q}'}_{\vec{\beta}}(\mathcal{D})$ to $L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$. □

In order to prove the conclusion of the Sect. 5 of this article, we introduce the following four lemmas about the Schur's test. All lemmas are derived from [10]. Therefore, we omit their proofs.

Lemma 3.5. *Let $\vec{\mu} := \mu_1 \times \mu_2$ and $\vec{v} := v_1 \times v_2$ be positive measures on the space $X \times X$ and, for $i \in \{1,2\}, K_i$ be nonnegative functions on $X \times X$. Let T be an integral operator with kernel $K := K_1 \cdot K_2$ defined by setting for any $(x,y) \in X \times X$,*

$$Tf(x,y) := \int_X \int_X K_1(x,s)K_2(y,t)f(s,t)d\mu_1(s)d\mu_2(t).$$

Suppose $\vec{p} := (p_1,p_2) \in (1,\infty) \times (1,\infty), \vec{q} := (q_1,q_2) \in (1,\infty) \times (1,\infty)$ satisfying $1 < p_- \le p_+ \le q_- < \infty$, where $p_+ := \max\{p_1,p_2\}, p_- := \min\{p_1,p_2\}$, and $q_- := \min\{q_1,q_2\}$. Let γ_i and δ_i be real numbers such that $\gamma_i + \delta_i = 1$ for $i \in \{1,2\}$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x,y) \in X \times X$

$$\begin{aligned} &\int_X \left(\int_X (K_1(x,s))^{\gamma_1 p_1'} (K_2(y,t))^{\gamma_2 p_1'} (h_1(s,t))^{p_1'} d\mu_1(s) \right)^{p_2'/p_1'} d\mu_2(t) \\ &\quad \le C_1 (h_2(x,y))^{p_2'} \end{aligned} \tag{3.1}$$

and for almost all $(s,t) \in X \times X$,

$$\begin{aligned} &\int_X \left(\int_X (K_1(x,s))^{\delta_1 q_1} (K_2(y,t))^{\delta_2 q_1} (h_2(x,y))^{q_1} d\nu_1(x) \right)^{q_2/q_1} dv_2(y) \\ &\quad \le C_2 (h_1(s,t))^{q_2}, \end{aligned} \tag{3.2}$$

then $T : L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}$ is bounded with $\|T\|_{L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}} \le C_1^{1/p_2'} C_2^{1/q_2}$.

Lemma 3.6. *Let $\vec{\mu}, \vec{v}$, the kernel K, and the operator T be as in Lemma 3.5. Suppose $\vec{q} := (q_1,q_2) \in [1,\infty) \times [1,\infty)$. Let γ_i and δ_i be two real numbers such that $\gamma_i + \delta_i = 1$ for $i \in \{1,2\}$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x,y) \in X \times X$,*

$$\operatorname*{ess\,sup}_{(s,t) \in X \times X} (K_1(x,s))^{\gamma_1} (K_2(y,t))^{\gamma_2} h_1(s,t) \le C_1 h_2(x,y) \tag{3.3}$$

and, for almost all $(s,t) \in X \times X$,

$$\begin{aligned} &\int_X \left(\int_X (K_1(x,s))^{\delta_1 q_1} (K_2(y,t))^{\delta_2 q_1} (h_2(x,y))^{q_1} d\nu_1(x) \right)^{q_2/q_1} d\nu_2(y) \\ &\quad \le C_2 (h_1(s,t))^{q_2}, \end{aligned} \tag{3.4}$$

then $T : L^{\vec{1}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}$ is bounded with $\|T\|_{L^{\vec{1}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}} \le C_1 C_2^{1/q_2}$.

Lemma 3.7. *Let $\vec{\mu}, \vec{v}$, the kernel K, and the operator T be as in Lemma 3.5. Suppose $\vec{p} = (p_1,1)$ with $p_1 \in (1,\infty)$ and $\vec{q} := (q_1,q_2) \in (1,\infty) \times (1,\infty)$ satisfying $1 < p_1 \le q_- < \infty$, where $q_- := \min\{q_1,q_2\}$. Let γ_i and δ_i be two real numbers such that $\gamma_i + \delta_i = 1$ for $i \in \{1,2\}$. If there exist two positive*

functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x, y) \in X \times X$,

$$\operatorname*{ess\,sup}_{t \in X} \int_X (K_1(x,s))^{\gamma_1 p_1'} (K_2(y,t))^{\gamma_2 p_1'} (h_1(s,t))^{p_1'} \, d\mu_1(s) \le C_1 (h_2(x,y))^{p_1'} \tag{3.5}$$

and, for almost all $(s, t) \in X \times X$,

$$\int_X \left(\int_X (K_1(x,s))^{\delta_1 q_1} (K_2(y,t))^{\delta_2 q_1} (h_2(x,y))^{q_1} \, d\nu_1(x) \right)^{q_2/q_1} d\nu_2(y) \le C_2 (h_1(s,t))^{q_2}, \tag{3.6}$$

then $T : L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}$ is bounded with $\|T\|_{L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}} \le C_1^{1/p_1'} C_2^{1/q_2}$

Lemma 3.8. *Let $\vec{\mu}, \vec{v}$, the kernel K, and the operator T be as in Lemma 3.5. Suppose $\vec{p} = (1, p_2)$ with $p_2 \in (1, \infty)$ and $\vec{q} := (q_1, q_2) \in (1, \infty) \times (1, \infty)$ satisfying $1 < p_2 \le q_- < \infty$, where $q_- := \min\{q_1, q_2\}$. Let γ_i and δ_i be two real numbers such that $\gamma_i + \delta_i = 1$ for $i \in \{1, 2\}$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x, y) \in X \times X$,*

$$\int_X \left(\operatorname*{ess\,sup}_{s \in X} (K_1(x,s))^{\gamma_1} (K_2(y,t))^{\gamma_2} h_1(s,t) \right)^{p_2'} d\mu_2(t) \le C_1 (h_2(x,y))^{p_2'} \tag{3.7}$$

and, for almost all $(s, t) \in X \times X$,

$$\int_X \left(\int_X (K_1(x,s))^{\delta_1 q_1} (K_2(y,t))^{\delta_2 q_1} (h_2(x,y))^{q_1} \, dv_1(x) \right)^{q_2/q_1} dv_2(y) \le C_2 (h_1(s,t))^{q_2}, \tag{3.8}$$

then $T : L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}$ is bounded with $\|T\|_{L^{\vec{p}}_{\vec{\mu}} \to L^{\vec{q}}_{\vec{v}}} \le C_1^{1/p_2'} C_2^{1/q_2}$.

Finally, we introduce a sufficient condition for the boundedness of an operator $S_{\vec{a},\, \vec{b},\, \vec{c}}$ when its codomain is $L^\infty(\mathcal{D})$.

Lemma 3.9. *Let $\vec{\mu} := \mu_1 \times \mu_2$ and $\vec{v} := v_1 \times v_2$ be positive measures on the space $X \times X$ and, for $i \in \{1, 2\}$, K_i be nonnegative functions on $X \times X$. Let T be an integral operator with kernel $K := K_1 \cdot K_2$ defined by setting for any $(x, y) \in X \times X$,*

$$Tf(x,y) := \int_X \int_X K_1(x,s) K_2(y,t) f(s,t) d\mu_1(s) d\mu_2(t).$$

Suppose $\vec{p} := (p_1, p_2), \vec{q} := (\infty, \infty)$ satisfying $1 \le p_- \le p_+ \le \infty$, if

$$\|K_1(x,\cdot) \cdot K_2(y,\cdot)\|_{L^{\vec{p}'}_{\vec{\mu}}} \tag{3.9}$$

is uniformly bounded, then T is bounded from $L^{\vec{p}}_{\vec{\mu}}$ to L^∞.

Proof. This result can be easily obtained from the following mixed norm Hölder inequality, which is as follows :

$$|Tf(x,y)| \le \|K_1(x,\cdot) \cdot K_2(y,\cdot)\|_{L^{\vec{p}'}_{\vec{\mu}}} \|f\|_{L^{\vec{p}}_{\vec{\mu}}}.$$

□

4. The Necessity for the Boundedness of $T_{\vec{a},\,\vec{b},\,\vec{c}}$

In this section, we give the necessary conditions for the boundedness of the operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$, and the conclusions are the following Lemmas 4.1–4.13.

Lemma 4.1. *Let $1 < p_1 \le q_1 < \infty$ and $1 < p_2 \le q_2 < \infty$. If the operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1,\ \alpha_i + 1 < p_i\,(b_i+1)\,, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n+1+\beta_i}{q_i} - \frac{n+1+\alpha_i}{p_i}.$$

Proof. By duality, the boundedness of $T_{\vec{a},\,\vec{b},\,\vec{c}}$ from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ implies the boundedness of $T^*_{\vec{a},\,\vec{b},\,\vec{c}}$ from $L^{\vec{q}'}_{\vec{\beta}}(\mathcal{D})$ to $L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$.

In order for Tf to be well-defined, then for any $f \in L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$, the following function on u and η must satisfy

$$\frac{\rho(z)^{a_1}\rho(u)^{b_1-\alpha_1}\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,u)^{c_1}\rho(w,\eta)^{c_2}} \in L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$$

for any $z,\ w \in T_B$,

that is

$$\begin{aligned} &\rho(z)^{a_1}\left(\int_{T_B}\frac{\rho(u)^{p_1'(b_1-\alpha_1)+\alpha_1}}{|\rho(z,u)|^{c_1p_1'}}dV(u)\right)^{\frac{1}{p_1'}} \\ &\quad\times\rho(w)^{a_2}\left(\int_{T_B}\frac{\rho(\eta)^{p_2'(b_2-\alpha_2)+\alpha_2}}{|\rho(w,\eta)|^{c_2p_2'}}dV(\eta)\right)^{\frac{1}{p_2'}} \\ &\quad<\infty. \end{aligned} \tag{4.1}$$

By Lemma 3.1, (4.1) is ture if and only if

$$\begin{cases} p_i'(b_i-\alpha_i)+\alpha_i > -1, \\ c_ip_i' > n+1+p_i'(b_i-\alpha_i)+\alpha_i \end{cases}$$

for any $i \in \{1,2\}$.

Summing up the two inequalities, we get $c_i > n/p_i' > 0$ for any $i \in \{1,2\}$.

For $u > 0$ and $\eta > 0$, we put

$$f_{u,\eta}(z,w) = \frac{\rho(z)^{t_1}\rho(w)^{t_2}}{\rho(z,u\mathbf{i})^{s_1}\rho(w,\eta\mathbf{i})^{s_2}},\ z,w \in T_B,$$

where s_1, s_2, t_1 and t_2 are real parameters satisfying conditions

$$\begin{cases} s_i > 0, \\ t_i > \max\left\{-\frac{\beta_i+1}{q_i'}, -a_i-\beta_i-1\right\}, \\ s_i - t_i > \max\left\{\frac{n+1+\beta_i}{q_i'}, \beta_i+a_i-c_i+n+1\right\}, \end{cases} \tag{4.2}$$

and $\mathbf{i}=(0',i)$.

By lemma 3.1, condition 4.2 makes $f_{u,\eta}(z,w)\in L^{\vec{q}'}_{\vec{\beta}}(\mathcal{D})$, and there exists a constant C such that

$$\|f_{u,\eta}(z,w)\|_{\vec{q}',\vec{\beta}}=Cu^{\frac{n+1+\beta_1}{q_1'}-(s_1-t_1)}\eta^{\frac{n+1+\beta_2}{q_2'}-(s_2-t_2)}.$$

By Lemmas 3.1 and 3.4, we have

$$\begin{aligned}
&T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\\
&\quad=\rho(z)^{b_1-\alpha_1}\rho(w)^{b_2-\alpha_2}\int_{T_B}\int_{T_B}\frac{\rho(l)^{\beta_1+a_1}\rho(m)^{\beta_2+a_2}}{\rho(z,l)^{c_1}\rho(w,m)^{c_2}}f_{u,\eta}(l,m)\,dV(l)dV(m)\\
&\quad=C\frac{\rho(z)^{b_1-\alpha_1}}{\rho(z,u\mathbf{i})^{c_1+s_1-(n+1+\beta_1+a_1+t_1)}}\cdot\frac{\rho(w)^{b_2-\alpha_2}}{\rho(w,\eta\mathbf{i})^{c_2+s_2-(n+1+\beta_2+a_2+t_2)}}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\|T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\|_{\vec{p}',\vec{\alpha}}\\
&\quad=C\left(\int_{T_B}\left(\int_{T_B}\left|T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\right|^{p_1'}dV_{\alpha_1}(z)\right)^{\frac{p_2'}{p_1'}}dV_{\alpha_2}(w)\right)^{\frac{1}{p_2'}}\\
&\quad=C\left(\int_{T_B}\frac{\rho(z)^{(b_1-\alpha_1)p_1'+\alpha_1}}{\rho(z,u\mathbf{i})^{(c_1+s_1-(\beta_1+t_1+a_1+n+1))p_1'}}dV(z)\right)^{\frac{1}{p_1'}}\\
&\qquad\times\left(\int_{T_B}\frac{\rho(w)^{(b_2-\alpha_2)p_2'+\alpha_2}}{\rho(w,\eta\mathbf{i})^{(c_2+s_2-(\beta_2+t_2+a_2+n+1))p_2'}}dV(w)\right)^{\frac{1}{p_2'}}.
\end{aligned}$$

Since $T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\in L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$, by Lemma 3.1 we have

$$(b_i-\alpha_i)\,p_i'+\alpha_i>-1,$$

that is

$$\alpha_i+1<p_i\,(b_i+1)\,.$$

Applying the above proof process to the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$, we can get

$$-q_ia_i<\beta_i+1$$

for any $i\in\{1,2\}$.

Next, we prove that for any $i\in\{1,2\}$, $c_i=n+b_i+1+\lambda_i$.

For any $\xi,\ \eta\in T_B$, let

$$f_{\xi,\eta}(z,w)=\frac{\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\rho(\eta)^{n+1+b_2-(n+1+\alpha_2)/p_2}}{\rho(z,\xi)^{n+1+b_1}\rho(w,\eta)^{n+1+b_2}}.$$

Given $\alpha_1+1<p_1(b_1+1)$ and $\alpha_2+1<p_2(b_2+1)$, it follows that there exists a positive constant C independent of ξ and η such that

$$\begin{aligned}\|f_{\xi,\eta}(z,w)\|_{\vec{p},\vec{\alpha}} &= \left(\int_{T_B}\left(\int_{T_B}|f_{\xi,\eta}(z,w)|^{p_1}\,dV_{\alpha_1}(z)\right)^{\frac{p_2}{p_1}}dV_{\alpha_2}(w)\right)^{\frac{1}{p_2}}\\ &= \left(\int_{T_B}\frac{\rho(\xi)^{(n+1+b_1)p_1-(n+1+\alpha_1)}\,\rho(z)^{\alpha_1}}{|\rho(z,\xi)|^{(n+1+b_1)p_1}}dV(z)\right)^{\frac{1}{p_1}}\\ &\quad\times\left(\int_{T_B}\frac{\rho(\eta)^{(n+1+b_2)p_2-(n+1+\alpha_2)}\,\rho(w)^{\alpha_2}}{|\rho(w,\eta)|^{(n+1+b_2)p_2}}dV(w)\right)^{\frac{1}{p_2}}\\ &\le C.\end{aligned}$$

Notice that $c_i>0$ and $b_i>(\alpha_i+1)/p_i-1>-1$, by Lemma 3.1 we have

$$\begin{aligned}&T_{\vec{a},\,\vec{b},\,\vec{c}}\,f_{\xi,\eta}(z,w)\\ &=\rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(l)^{b_1}\,\rho(m)^{b_2}}{\rho(z,l)^{c_1}\,\rho(w,m)^{c_2}}f_{\xi,\eta}(l,m)\,dV(l)dV(m)\\ &=\rho(z)^{a_1}\rho(w)^{a_2}\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\,\rho(\eta)^{n+1+b_2-(n+1+\alpha_2)/p_2}\\ &\quad\times\int_{T_B}\frac{\rho(l)^{b_1}}{\rho(z,l)^{c_1}\,\rho(l,\xi)^{n+1+b_1}}dV(l)\int_{T_B}\frac{\rho(m)^{b_2}}{\rho(w,m)^{c_2}\,\rho(m,\eta)^{n+1+b_2}}dV(m)\\ &=C\frac{\rho(z)^{a_1}\rho(w)^{a_2}\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\,\rho(\eta)^{n+1+b_2-(n+1+\alpha_2)/p_2}}{\rho(z,\xi)^{c_1}\,\rho(w,\eta)^{c_2}}.\end{aligned}$$

Since $T_{\vec{a},\,\vec{b},\,\vec{c}}\,f_{\xi,\eta}(z,w)\in L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, there is a positive constant C such that

$$\begin{aligned}&\|T_{\vec{a},\,\vec{b},\,\vec{c}}\,f_{\xi,\eta}(z,w)\|_{\vec{q},\vec{\beta}}\\ &=\left(\int_{T_B}\left(\int_{T_B}\left|T_{\vec{a},\,\vec{b},\,\vec{c}}\,f_{\xi,\eta}(z,w)\right|^{q_1}dV_{\beta_1}(z)\right)^{\frac{q_2}{q_1}}dV_{\beta_2}(w)\right)^{\frac{1}{q_2}}\\ &=\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\,\rho(\eta)^{n+1+b_2-(n+1+\alpha_2)/p_2}\\ &\quad\times\left(\int_{T_B}\frac{\rho(z)^{\beta_1+a_1q_1}}{|\rho(z,\xi)|^{c_1q_1}}dV(z)\right)^{\frac{1}{q_1}}\left(\int_{T_B}\frac{\rho(w)^{\beta_2+a_2q_2}}{|\rho(w,\eta)|^{c_2q_2}}dV(w)\right)^{\frac{1}{q_2}}\\ &\le C.\end{aligned}$$

By Lemma 3.1, for any $i\in\{1,2\}$, we obtain

$$n+1+b_i-(n+1+\alpha_i)/p_i=(c_iq_i-(n+1+\beta_i+a_iq_i))/q_i$$

which simplifies to

$$c_i=n+b_i+1+\lambda_i.$$

□

Lemma 4.2. *Let $1 = p_1 \le q_1 < \infty$ and $1 = p_2 \le q_2 < \infty$. If the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1,\ \alpha_i < b_i, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n+1+\beta_i}{q_i} - (n+1+\alpha_i).$$

Proof. Similar to the proof of Lemma 4.1.

We have

$$\begin{aligned} &T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w) \\ &\quad = C \frac{\rho(z)^{b_1-\alpha_1}}{\rho(z,u\mathbf{i})^{c_1+s_1-(n+1+\beta_1+a_1+t_1)}} \cdot \frac{\rho(w)^{b_2-\alpha_2}}{\rho(w,\eta\mathbf{i})^{c_2+s_2-(n+1+\beta_2+a_2+t_2)}}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \|T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\|_\infty = C &\operatorname*{ess\,sup}_{z\in T_B} \frac{\rho(z)^{b_1-\alpha_1}}{\rho(z,u\mathbf{i})^{c_1+s_1-(\beta_1+t_1+a_1+n+1)}} \\ &\cdot \operatorname*{ess\,sup}_{w\in T_B} \frac{\rho(w)^{b_2-\alpha_2}}{\rho(w,\eta\mathbf{i})^{c_2+s_2-(\beta_2+t_2+a_2+n+1)}}. \end{aligned} \tag{4.3}$$

Since $T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w) \in L^\infty(\mathcal{D})$, the above equation is finite. According to the conditions satisfied by s_i and t_i, it can be obtained that the power of the denominator of the above formula (4.3) is greater than zero, and $\rho(z)$, $\rho(w)$ can be infinite on T_B. Therefore, combined with the lemma 3.2, we obtain $b_1 > \alpha_1$ and $b_2 > \alpha_2$.

Applying the above proof process to the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$, we can get

$$-q_i a_i < \beta_i + 1$$

for any $i \in \{1,2\}$.

Next, we prove that for any $i \in \{1,2\}$, $c_i = n + b_i + 1 + \lambda_i$.

For any $\xi,\ \eta \in T_B$, let

$$f_{\xi,\eta}(z,w) = \frac{\rho(\xi)^{b_1-\alpha_1}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,\xi)^{n+1+b_1}\rho(w,\eta)^{n+1+b_2}}.$$

Given $\alpha_1 < b_1$ and $\alpha_2 < b_2$, we can easily see that there is a positive constant C independent of ξ and η such that

$$\begin{aligned} \|f_{\xi,\eta}(z,w)\|_{\vec{1},\vec{\alpha}} &= \int_{T_B}\int_{T_B} |f_{\xi,\eta}(z,w)|\, dV_{\alpha_1}(z) dV_{\alpha_2}(w) \\ &= \int_{T_B} \frac{\rho(\xi)^{b_1-\alpha_1}\rho(z)^{\alpha_1}}{|\rho(z,\xi)|^{n+1+b_1}} dV(z) \int_{T_B} \frac{\rho(\eta)^{b_2-\alpha_2}\rho(w)^{\alpha_2}}{|\rho(w,\eta)|^{n+1+b_2}} dV(w) \\ &\le C \end{aligned}$$

Notice that $c_i > 0$ and $b_i > \alpha_i > -1$, by Lemma 3.1 we have

$$\begin{aligned}
&T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\\
&= \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(l)^{b_1}\,\rho(m)^{b_2}}{\rho(z,l)^{c_1}\,\rho(w,m)^{c_2}}f_{\xi,\eta}(l,m)\,dV(l)dV(m)\\
&= \rho(z)^{a_1}\rho(\xi)^{b_1-\alpha_1}\int_{T_B}\frac{\rho(l)^{b_1}}{\rho(z,l)^{c_1}\,\rho(l,\xi)^{n+1+b_1}}dV(l)\\
&\quad\times\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}\int_{T_B}\frac{\rho(m)^{b_2}}{\rho(w,m)^{c_2}\,\rho(m,\eta)^{n+1+b_2}}dV(m)\\
&= C\frac{\rho(z)^{a_1}\rho(w)^{a_2}\rho(\xi)^{b_1-\alpha_1}\,\rho(\eta)^{b_2-\alpha_2}}{\rho(z,\xi)^{c_1}\,\rho(w,\eta)^{c_2}}.
\end{aligned}$$

Since $T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w) \in L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, we know that there is a positive constant C such that

$$\begin{aligned}
&\left\|T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\right\|_{\vec{q},\vec{\beta}}\\
&= \left(\int_{T_B}\left(\int_{T_B}\left|T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\right|^{q_1}dV_{\beta_1}(z)\right)^{\frac{q_2}{q_1}}dV_{\beta_2}(w)\right)^{\frac{1}{q_2}}\\
&= \rho(\xi)^{b_1-\alpha_1}\left(\int_{T_B}\frac{\rho(z)^{a_1q_1+\beta_1}}{|\rho(z,\xi)|^{c_1q_1}}dV(z)\right)^{\frac{1}{q_1}}\\
&\quad\times\rho(\eta)^{b_2-\alpha_2}\left(\int_{T_B}\frac{\rho(w)^{a_2q_2+\beta_2}}{|\rho(w,\eta)|^{c_2q_2}}dV(w)\right)^{\frac{1}{q_2}}\\
&\le C.
\end{aligned}$$

By Lemma 3.1, for any $i \in \{1,2\}$, we have

$$b_i - \alpha_i = \left(c_iq_i - (n+1+a_iq_i+\beta_i)\right)/q_i,$$

which simplifies to

$$c_i = n + b_i + 1 + \lambda_i,$$

where

$$\lambda_i = \frac{n+1+\beta_i}{q_i} - (n+1+\alpha_i).$$

□

Lemma 4.3. *Let $1 < p_1 \le q_1 < \infty$ and $1 = p_2 \le q_2 < \infty$. If the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases}\alpha_1 + 1 < p_1(b_1+1),\ \alpha_2 < b_2,\\ -q_ia_i < \beta_i + 1,\ c_i = n+1+a_i+b_i+\lambda_i,\end{cases}$$

where

$$\begin{cases}\lambda_1 = \frac{n+1+\beta_1}{q_1} - \frac{n+1+\alpha_1}{p_1},\\ \lambda_2 = \frac{n+1+\beta_2}{q_2} - (n+1+\alpha_2).\end{cases}$$

Proof. Similar to the proof of Lemma 4.1. We get

$$T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w) = C\frac{\rho(z)^{b_1-\alpha_1}}{\rho(z,u\mathbf{i})^{c_1+s_1-(n+1+\beta_1+a_1+t_1)}}\cdot\frac{\rho(w)^{b_2-\alpha_2}}{\rho(w,\eta\mathbf{i})^{c_2+s_2-(n+1+\beta_2+a_2+t_2)}}.$$

It is easy to see that

$$\begin{aligned}\left\|T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\right\|_{\vec{p}',\vec{\alpha}} &= C\operatorname*{ess\,sup}_{w\in T_B}\left(\int_{T_B}\left|T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\right|^{p_1'}dV_{\alpha_1}(z)\right)^{\frac{1}{p_1'}}\\ &\quad\times\left(\int_{T_B}\frac{\rho(z)^{(b_1-\alpha_1)p_1'+\alpha_1}}{\rho(z,u\mathbf{i})^{(c_1+s_1-(\beta_1+t_1+a_1+n+1))p_1'}}dV(z)\right)^{\frac{1}{p_1'}}\\ &= C\operatorname*{ess\,sup}_{w\in T_B}\frac{\rho(w)^{b_2-\alpha_2}}{\rho(w,\eta\mathbf{i})^{c_2+s_2-(\beta_2+t_2+a_2+n+1)}}.\end{aligned}$$

Since $T^*_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{u,\eta}(z,w)\in L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D})$, by Lemma 3.1 we have

$$(b_1-\alpha_1)\,p_1'+\alpha_1>-1,$$

that is

$$\alpha_1+1<p_1(b_1+1).$$

According to $s_2-t_2>\beta_2+a_2-c_2+n+1$, the power of the denominator of $\rho(w)^{b_2-\alpha_2}/\rho(w,\eta\mathbf{i})^{c_2+s_2-(\beta_2+t_2+n+1)}$ is greater than 0, and $\rho(w)$ can be infinite on T_B. Combined with Lemma 3.2, the above finite means $b_2>\alpha_2$.

Applying the above proof process to the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$, we can get

$$-q_ia_i<\beta_i+1$$

for any $i\in\{1,2\}$.

Next, we prove that for any $i\in\{1,2\}$, $c_i=n+b_i+1+\lambda_i$.

For any $\xi,\ \eta\in T_B$, let

$$f_{\xi,\eta}(z,w)=\frac{\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,\xi)^{n+1+b_1}\rho(w,\eta)^{n+1+b_2}}.$$

Given $\alpha_1+1<p_1(b_1+1)$ and $\alpha_2<b_2$, we can easily see that there is a positive constant C independent of ξ and η such that

$$\begin{aligned}\left\|f_{\xi,\eta}(z,w)\right\|_{\vec{p},\vec{\alpha}} &= \int_{T_B}\left(\int_{T_B}\left|f_{\xi,\eta}(z,w)\right|^{p_1}dV_{\alpha_1}(z)\right)^{\frac{1}{p_1}}dV_{\alpha_2}(w)\\ &= \int_{T_B}\frac{\rho(\xi)^{(n+1+b_1)p_1-(n+1+\alpha_1)}\rho(z)^{\alpha_1}}{\left|\rho(z,\xi)\right|^{(n+1+b_1)p_1}}dV(z)\\ &\quad\times\int_{T_B}\frac{\rho(\eta)^{b_2-\alpha_2}\rho(w)^{\alpha_2}}{\left|\rho(w,\eta)\right|^{n+1+b_2}}dV(w)\\ &\le C.\end{aligned}$$

Notice that $c_i > 0$, $b_1 > (\alpha+1)/p - 1 > -1$ and $b_2 > \alpha_2 > -1$, by Lemma 3.1 we have

$$\begin{aligned}
&T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\\
&\quad = \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(l)^{b_1}\rho(m)^{b_2}}{\rho(z,l)^{c_1}\rho(w,m)^{c_2}} f_{\xi,\eta}(l,m)\,dV(l)dV(m)\\
&\quad = C\frac{\rho(z)^{a_1}\rho(w)^{a_2}\rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,\xi)^{c_1}\rho(w,\eta)^{c_2}}.
\end{aligned}$$

Since $T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w) \in L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, we know that there is a positive constant C such that

$$\begin{aligned}
&\|T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\|_{\vec{q},\vec{\beta}}\\
&\quad = \left(\int_{T_B}\left(\int_{T_B}\left|T_{\vec{a},\ \vec{b},\ \vec{c}}\, f_{\xi,\eta}(z,w)\right|^{q_1} dV_{\beta_1}(z)\right)^{\frac{q_2}{q_1}} dV_{\beta_2}(w)\right)^{\frac{1}{q_2}}\\
&\quad = \rho(\xi)^{n+1+b_1-(n+1+\alpha_1)/p_1}\rho(\eta)^{b_2-\alpha_2}\\
&\qquad \times\left(\int_{T_B}\frac{\rho(z)^{a_1q_1+\beta_1}}{|\rho(z,\xi)|^{c_1q_1}}dV(z)\right)^{\frac{1}{q_1}}\left(\int_{T_B}\frac{\rho(w)^{a_2q_2+\beta_2}}{|\rho(w,\eta)|^{c_2q_2}}dV(w)\right)^{\frac{1}{q_2}}\\
&\quad \le C.
\end{aligned}$$

Hence, by Lemma 3.1 we have

$$n+1+b_1-(n+1+\alpha_1)/p_1 = (c_1q_1-(n+1+a_1q_1+\beta_1))/q_1$$

and

$$b_2-\alpha_2 = (c_2q_2-(n+1+a_2q_2+\beta_2))/q_2,$$

that is,

$$c_1 = n+1+a_1+b_1+\frac{n+1+\beta_1}{q_1}-\frac{n+1+\alpha_1}{p_1}$$

and

$$c_2 = a_2+b_2-\alpha_2+\frac{n+1+\beta_2}{q_2}.$$

□

Lemma 4.4. *Let $1 = p_1 \le q_1 < \infty$ and $1 < p_2 \le q_2 < \infty$. If the operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases}\alpha_1 < b_1,\ \alpha_2+1 < p_2(b_2+1),\\ -q_ia_i < \beta_i+1,\ c_i = n+1+a_i+b_i+\lambda_i,\end{cases}$$

where

$$\begin{cases}\lambda_1 = \frac{n+1+\beta_1}{q_1}-(n+1+\alpha_1),\\ \lambda_2 = \frac{n+1+\beta_2}{q_2}-\frac{n+1+\alpha_2}{p_2}.\end{cases}$$

Proof. The proof of Lemma 4.4 is similar to the proof of Lemma 4.3, which only needs to modify the corresponding norm appropriately, thus we omit its proof. □

Lemma 4.5. *Let* $1 < p_1 < q_1 = \infty$ *and* $1 < p_2 < q_2 = \infty$. *Suppose* $T_{\vec{a},\ \vec{b},\ \vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\infty}(\mathcal{D})$, *then for any* $i \in \{1,2\}$,

$$\begin{cases} a_i > 0,\ \alpha_i + 1 < p_i(b_i+1), \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\lambda_i = -\frac{n+1+\alpha_i}{p_i}.$$

Proof. Since $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded, it follows that for any fixed $z, w \in T_B$, the integral

$$\begin{aligned} &T_{\vec{a},\ \vec{b},\ \vec{c}}\, f(z,w) \\ &= \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(u)^{b_1}\,\rho(\eta)^{b_2}}{\rho(z,u)^{c_1}\,\rho(w,\eta)^{c_2}} f(u,\eta)\,dV(u)dV(\eta) \\ &= \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(u)^{b_1-\alpha_1}\,\rho(\eta)^{b_2-\alpha_2}}{\rho(z,u)^{c_1}\,\rho(w,\eta)^{c_2}} f(u,\eta)\,dV_{\alpha_1}(u)dV_{\alpha_2}(\eta) \end{aligned}$$

is finte for each $f \in L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$.

By duality, we get that for any fixed $z, w \in T_B$, the function on variables u and η must satisfy

$$\frac{\rho(z)^{a_1}\rho(u)^{b_1-\alpha_1}\,\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,u)^{c_1}\,\rho(w,\eta)^{c_2}} \in L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D}).$$

Therefore, by calculation, we have

$$\begin{aligned} &\rho(z)^{a_1}\left(\int_{T_B}\frac{\rho(u)^{p_1'(b_1-\alpha_1)+\alpha_1}}{|\rho(z,u)|^{c_1p_1'}}dV(u)\right)^{\frac{1}{p_1'}} \\ &\quad\times\rho(w)^{a_2}\left(\int_{T_B}\frac{\rho(\eta)^{p_2'(b_2-\alpha_2)+\alpha_2}}{|\rho(w,\eta)|^{c_2p_2'}}dV(\eta)\right)^{\frac{1}{p_2'}} \\ &\lesssim \frac{\rho(z)^{a_1}\,\rho(w)^{a2}}{\rho(z)^{c_1+\alpha_1-b_1-(n+1+\alpha_1)/p_1'}\,\rho(w)^{c_2+\alpha_2-b_2-(n+1+\alpha_2)/p_2'}} \\ &< \infty. \end{aligned}$$

By Lemma 3.1, we know that for any $i \in \{1,2\}$,

$$\begin{cases} p_i'(b_i-\alpha_i)+\alpha_i > -1, \\ c_ip_i' > n+1+p_i'(b_i-\alpha_i)+\alpha_i, \end{cases}$$

that is

$$\begin{cases} \alpha_i+1 < p_i(b_i+1), \\ c_i > n+1+b_i-\frac{n+1+\alpha_i}{p_i}. \end{cases}$$

By the arbitrariness of z and w, we have

$$\begin{cases} a_i > 0, \\ c_i = n+1+a_i+b_i-\frac{n+1+\alpha_i}{p_i}. \end{cases}$$

This completes the proof of the lemma. □

Lemma 4.6. *Let $1=p_1<q_1=\infty$ and $1=p_2<q_2=\infty$. Suppose $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i\in\{1,2\}$,*

$$\begin{cases} a_i \geq 0,\ b_i \geq \alpha_i, \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\lambda_i = -n-1-\alpha_i.$$

Proof. Similar to the proof of Lemma 4.5, by duality, we obtain that for any fixed $z,w\in T_B$, the function on variables u and η must satisfy

$$\operatorname*{ess\,sup}_{(u,\eta)\in\mathcal{D}} \frac{\rho(z)^{a_1}\rho(u)^{b_1-\alpha_1}\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}}{|\rho(z,u)|^{c_1}|\rho(w,\eta)|^{c_2}} < \infty.$$

According to Lemma 3.2 and combining the arbitrariness of z and w, we can easily prove this result. □

Lemma 4.7. *Let $1=p_1<q_1=\infty$ and $1<p_2<q_2=\infty$. Suppose $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i\in\{1,2\}$,*

$$\begin{cases} a_1 \geq 0,\ a_2 > 0, \\ b_1 \geq \alpha_1,\ p_2(b_2+1) > \alpha_2+1, \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -n-1-\alpha_1, \\ \lambda_2 = -\frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Proof. Similar to the method of lemma 4.5, by duality, we have for any fixed $z,w\in T_B$, the function on variables u and η must satisfy

$$\frac{\rho(z)^{a_1}\rho(u)^{b_1-\alpha_1}\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}}{\rho(z,u)^{c_1}\rho(w,\eta)^{c_2}} \in L^{\vec{p}'}_{\vec{\alpha}}(\mathcal{D}),$$

that is

$$\rho(z)^{a_1}\rho(w)^{a_2}\operatorname*{ess\,sup}_{u\in T_B}\frac{\rho(u)^{b_1-\alpha_1}}{|\rho(z,u)|^{c_1}}\left(\int_{T_B}\frac{\rho(\eta)^{p_2'(b_2-\alpha_2)+\alpha_2}}{|\rho(w,\eta)|^{c_2p_2'}}dV(\eta)\right)^{\frac{1}{p_2'}} < \infty.$$

Similar to the method of Lemma 4.5 and Lemma 4.6, we can complete the proof of the lemma. □

Lemma 4.8. *Let $1 < p_1 < q_1 = \infty$ and $1 = p_2 < q_2 = \infty$. Suppose $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_1 > 0,\ a_2 \geq 0, \\ p_1 (b_1 + 1) > \alpha_1 + 1,\ b_2 \geq \alpha_2, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -\frac{n+1+\alpha_1}{p_1}, \\ \lambda_2 = -n - 1 - \alpha_2. \end{cases}$$

Proof. Lemma 4.8 is the symmetric case of Lemma 4.7. Thus, we omit the proof here. □

Lemma 4.9. *Let $p_1 = q_1 = \infty$ and $p_2 = q_2 = \infty$. Suppose $p_2 = q_2 = \infty$ is bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_i > 0,\ b_i > -1, \\ c_i = n + 1 + a_i + b_i. \end{cases}$$

Proof. For any $f \in L^{\infty}(\mathcal{D})$, the boundedness of $T_{\vec{a},\ \vec{b},\ \vec{c}} : L^{\infty}(\mathcal{D}) \to L^{\infty}(\mathcal{D})$ implies that

$$\rho(z)^{a_1} \rho(w)^{a_2} \int_{T_B} \int_{T_B} \frac{\rho(u)^{b_1} \rho(\eta)^{b_2}}{|\rho(z,u)|^{c_1} |\rho(w,\eta)|^{c_2}} dV(u) dV(\eta) < \infty, \quad (4.4)$$

for any fixed $z, w \in T_B$.

It follows from Lemma 3.1 that $b_i > -1$ and $c_i > n + 1 + b_i$ for any $i \in \{1,2\}$. (4.4) becomes as follows :

$$\frac{\rho(z)^{a_1} \rho(w)^{a_2}}{\rho(z)^{c_1-(b_1+n+1)} \rho(w)^{c_2-(b_2+n+1)}} < \infty.$$

By the arbitrariness of z and w, this means $a_i > 0$ and $c_i = n+1+a_i+b_i$ for any $i \in \{1,2\}$. □

Lemma 4.10. *Let $1 = p_1 < q_1 = \infty$ and $p_2 = q_2 = \infty$. Suppose $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_1 \geq 0,\ a_2 > 0, \\ b_1 \geq \alpha_1,\ b_2 > -1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -n - 1 - \alpha_1, \\ \lambda_2 = 0. \end{cases}$$

Proof. Since $p_1 = 1$, $p_2 = \infty$, we can conclude that for any $f \in L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$, there exists a positive constant C and a function g such that $|f(z,w)| \le C\,|g(z)|$, where g satisfies

$$\int_{T_B} |g(z)|\, dV_{\alpha_1}(z) < \infty.$$

Since $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded, it follows that for any fixed $z, w \in T_B$, the integral

$$\begin{aligned}
&\left|T_{\vec{a},\,\vec{b},\,\vec{c}}\, f(z,w)\right| \\
&\le \rho(z)^{a_1}\rho(w)^{a_2}\int_{T_B}\int_{T_B}\frac{\rho(u)^{b_1-\alpha_1}\rho(\eta)^{b_2-\alpha_2}}{|\rho(z,u)|^{c_1}\,|\rho(w,\eta)|^{c_2}}\,|g(u)|\,dV_{\alpha_1}(u)dV_{\alpha_2}(\eta) \\
&= \rho(z)^{a_1}\int_{T_B}\frac{\rho(u)^{b_1-\alpha_1}}{|\rho(z,u)|^{c_1}}\,|g(u)|\,dV_{\alpha_1}(u) \\
&\times \rho(w)^{a_2}\int_{T_B}\frac{\rho(\eta)^{b_2-\alpha_2}}{|\rho(w,\eta)|^{c_2}}dV_{\alpha_2}(\eta)
\end{aligned} \tag{4.5}$$

is finte for each $f \in L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$.

The proof of the remaining part is similar to the proof of Lemmas 4.6 and 4.9, and we will not repeat it. □

Lemma 4.11. *Let $p_1 = q_1 = \infty$ and $1 = p_2 < q_2 = \infty$. Suppose $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_1 > 0,\ a_2 \ge 0, \\ b_1 > -1,\ b_2 \ge \alpha_2, \\ c_i = n + 1 + a_i + b_i + \lambda_i. \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -n - 1 - \alpha_2. \end{cases}$$

Proof. Lemma 4.11 is the symmetric case of Lemma 4.10. Thus, we omit the proof here. □

Lemma 4.12. *Let $p_1 = q_1 = \infty$ and $1 < p_2 < q_2 = \infty$. Suppose $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_1 > 0,\ a_2 > 0, \\ b_1 > -1,\ p_2(b_2+1) > \alpha_2 + 1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -\frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Proof. The proof of this lemma is similar to Lemma 4.10. From the known exponential condition, we know that there is a positive constant C and function g such that $|f(z,w)| \le Cg(w)$, where g satisfies the following equation:

$$\left(\int_{T_B} |g(w)|^{p_2} \, dV_{\alpha_2}(z)\right)^{\frac{1}{p_2}} < \infty.$$

The boundedness of $T_{\vec{a},\,\vec{b},\,\vec{c}} : L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D}) \to L^{\infty}(\mathcal{D})$ implies that

$$\begin{aligned}
&\left|T_{\vec{a},\,\vec{b},\,\vec{c}} f(z,w)\right| \\
&\quad \le \rho(z)^{a_1} \rho(w)^{a_2} \int_{T_B}\int_{T_B} \frac{\rho(u)^{b_1} \rho(\eta)^{b_2}}{|\rho(z,u)|^{c_1} |\rho(w,\eta)|^{c_2}} |g(\eta)| \, dV(u) dV(\eta) \\
&\quad = \rho(z)^{a_1} \int_{T_B} \frac{\rho(u)^{b_1}}{|\rho(z,u)|^{c_1}} dV(u) \\
&\qquad \times \rho(w)^{a_2} \int_{T_B} \frac{\rho(\eta)^{b_2-\alpha_2}}{|\rho(w,\eta)|^{c_2}} |g(\eta)| \, dV_{\alpha_2}(\eta)
\end{aligned}$$

Similar to the method of Lemma 4.9 and Lemma 4.5, and we will not repeat it. □

Lemma 4.13. *Let $1 < p_1 < q_1 = \infty$ and $p_2 = q_2 = \infty$. Suppose $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$, then for any $i \in \{1,2\}$,*

$$\begin{cases} a_1 > 0, \ a_2 > 0, \\ p_1(b_1+1) > \alpha_1 + 1, \ b_2 > -1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -\frac{n+1+\alpha_2}{p_2}, \\ \lambda_2 = 0. \end{cases}$$

Proof. Lemma 4.13 is the symmetric case of Lemma 4.12. Thus, we omit the proof here. □

5. The Sufficiency for the Boundedness of $S_{\vec{a},\,\vec{b},\,\vec{c}}$

In this section, we give the necessary conditions for the boundedness of the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$, and the conclusions are the following lemmas 5.1 to lemma 5.13.

Lemma 5.1. *Let $1 < p_- \le p_+ \le q_- < \infty$. If the parameters satisfy for any $i \in \{1,2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1, \ \alpha_i + 1 < p_i(b_i+1), \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n+1+\beta_i}{q_i} - \frac{n+1+\alpha_i}{p_i}.$$

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.

Proof. For any $i \in \{1,2\}$, let $\tau_i = c_i - a_i - b_i + \alpha_i = \frac{n+1+\alpha_i}{p_i'} + \frac{n+1+\beta_i}{q_i} > 0$.

Since $-q_i a_i < \beta_i + 1$, we have

$$-\frac{\tau_i\,(\beta_i+1)}{q_i} < a_i\tau_i,$$

which is equivalent to

$$-\frac{\tau_i\,(\beta_i+1)}{q_i} - \frac{a_i\,(n+1+\beta_i)}{q_i} < \frac{a_i\,(n+1+\alpha_i)}{p_i'}. \tag{5.1}$$

In addition, from $\alpha_i + 1 < p_i\,(b_i+1)$, it follows that

$$b_i - \alpha_i + \frac{\alpha_i+1}{p_i'} > 0.$$

Hence, we obtain

$$-\frac{(\alpha_i+1)\,\tau_i}{p_i'} - \frac{(b_i-\alpha_i)\,(n+1+\alpha_i)}{p_i'} < \frac{(b_i-\alpha_i)\,(n+1+\beta_i)}{q_i}. \tag{5.2}$$

By (5.1) and (5.2), there exist r_i and s_i such that

$$-\frac{\tau_i\,(\beta_i+1)}{q_i} - \frac{a_i\,(n+1+\beta_i)}{q_i} < r_i\tau_i + a_i\,(r_i - s_i) < \frac{a_i\,(n+1+\alpha_i)}{p_i'}$$

and

$$\begin{cases} -\frac{\tau_i(\alpha_i+1)}{p_i'} - \frac{(b_i-\alpha_i)(n+1+\alpha_i)}{p_i'} < s_i\tau_i + (b_i-\alpha_i)\,(s_i-r_i) \\ s_i\tau_i + (b_i-\alpha_i)\,(s_i-r_i) < \frac{(b_i-\alpha_i)(n+1+\beta_i)}{q_i} \end{cases},$$

which is equivalent to

$$-\frac{\beta_i+1}{q_i} - a_i\delta_i < r_i < a_i\gamma_i \tag{5.3}$$

and

$$-\frac{\alpha_i+1}{p_i'} - (b_i-\alpha_i)\,\gamma_i < s_i < (b_i-\alpha_i)\,\delta_i, \tag{5.4}$$

where

$$\gamma_i = \frac{(n+1+\alpha_i)\,/p_i' + s_i - r_i}{\tau_i}$$

and

$$\delta_i = \frac{(n+1+\beta_i)\,/q_i + r_i - s_i}{\tau_i}.$$

Obviously, for any $i \in \{1,2\}$, $\gamma_i + \delta_i = 1$.

Let $h_1(u,\eta) = \rho(u)^{s_1}\,\rho(\eta)^{s_2}$, $h_2(z,w) = \rho(z)^{r_1}\,\rho(w)^{r_2}$, write $S_{\vec{a},\,\vec{b},\,\vec{c}}$ as

$$S_{\vec{a},\,\vec{b},\,\vec{c}}\,f(z,w) = \int_{T_B}\int_{T_B} K_1(z,u)\,K_2(w,\eta)\,f(u,\eta)\,dV_{\alpha_1}(u)\,dV_{\alpha_2}(\eta),$$

where

$$K_1(z,u)=\frac{\rho(u)^{b_1-\alpha_1}\rho(z)^{a_1}}{|\rho(z,u)|^{c_1}},\ \text{and}\ K_2(w,\eta)=\frac{\rho(\eta)^{b_2-\alpha_2}\rho(w)^{a_2}}{|\rho(w,\eta)|^{c_2}}.$$

Now we prove this conclusion by lemma 3.5. We consider

$$\begin{aligned}&\int_{T_B}\left(K_1(z,u)\right)^{\gamma_1 p_1'}\left(K_2(w,\eta)\right)^{\gamma_2 p_1'}\left(h_1(u,\eta)\right)^{p_1'}dV_{\alpha_1}(u)\\&=\frac{\rho(z)^{a_1\gamma_1 p_1'}\rho(w)^{a_2\gamma_2 p_2'}\rho(\eta)^{(b_2-\alpha_2)\gamma_2 p_1'+s_2 p_1'}}{|\rho(w,\eta)|^{c_2\gamma_2 p_1'}}\\&\quad\times\int_{T_B}\frac{\rho(u)^{(b_1-\alpha_1)\gamma_1 p_1'+s_1 p_1'+\alpha_1}}{|\rho(z,u)|^{c_1\gamma_1 p_1'}}dV(u).\end{aligned}\tag{5.5}$$

From the first inequality in (5.4), we have

$$(b_i-\alpha_i)\gamma_i p_i'+s_i p_i'+\alpha_i>-1.\tag{5.6}$$

Notice that

$$(c_i-a_i-b_i+\alpha_i)\gamma_i=\tau_i\gamma_i=\frac{n+1+\alpha_i}{p_i'}+s_i-r_i,$$

by the second inequality in (5.3), we have

$$c_i\gamma_i p_i'-n-1-(b_i-\alpha_i)\gamma_i p_i'-s_i p_i'-\alpha_i=a_i\gamma_i p_i'-r_i p_i'>0.\tag{5.7}$$

Thus, according to Lemma 3.1, for any given $z\in T_B$, we have

$$\int_{T_B}\frac{\rho(u)^{(b_1-\alpha_1)\gamma_1 p_1'+s_1 p_1'+\alpha_1}}{|\rho(z,u)|^{c_1\gamma_1 p_1'}}dV(u)\lesssim\rho(z)^{r_1 p_1'-a_1\gamma_1 p_1'},$$

which, together with (5.5), (5.6), (5.7), and Lemma 3.1, we get

$$\begin{aligned}&\int_{T_B}\left(\int_{T_B}\left(K_1(z,u)\right)^{\gamma_1 p_1'}\left(K_2(w,\eta)\right)^{\gamma_2 p_1'}\left(h_1(u,\eta)\right)^{p_1'}dV_{\alpha_1}(u)\right)^{\frac{p_2'}{p_1'}}dV_{\alpha_2}(\eta)\\&\lesssim\rho(z)^{r_1 p_2'}\rho(w)^{a_2\gamma_2 p_2'}\int_{T_B}\frac{\rho(\eta)^{(b_2-\alpha_2)\gamma_2 p_2'+s_2 p_2'+\alpha_2}}{|\rho(w,\eta)|^{c_2\gamma_2 p_2'}}dV(\eta)\\&\lesssim\rho(z)^{r_1 p_2'}\rho(w)^{r_2 p_2'}=h_2(z,w)^{p_2'}.\end{aligned}$$

Thus, condition (3.1) holds ture for the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$.

Next, we verify condition (3.2). Notice that

$$\begin{aligned}&\int_{T_B}\left(K_1(z,u)\right)^{\delta_1 q_1}\left(K_2(w,\eta)\right)^{\delta_2 q_1}\left(h_2(z,w)\right)^{q_1}dV_{\beta_1}(z)\\&=\frac{\rho(u)^{(b_1-\alpha_1)\delta_1 q_1}\rho(\eta)^{(b_2-\alpha_2)\delta_2 q_1}\rho(w)^{r_2 q_1+a_2\delta_2 q_1}}{|\rho(w,\eta)|^{c_2\delta_2 q_1}}\\&\quad\times\int_{T_B}\frac{\rho(z)^{r_1 q_1+a_1\delta_1 q_1+\beta_1}}{|\rho(z,u)|^{c_1\delta_1 q_1}}dV(z).\end{aligned}$$

From the first inequality in (5.3), we have

$$r_i q_i+a_i\delta_i q_i+\beta_i>-1.\tag{5.8}$$

Notice that

$$(c_i - a_i - b_i + \alpha_i)\,\delta_i = \tau_i\delta_i = \frac{n+1+\beta_i}{q_i} + r_i - s_i,$$

by the second inequality in (5.4), we have

$$c_i\delta_i q_i - n - 1 - r_i q_i - a_i\delta_i q_i - \beta_i = (b_i - \alpha_i)\,\delta_i q_i - s_i q_i > 0. \tag{5.9}$$

Thus, according to Lemma 3.1, for any given $u \in T_B$, we have

$$\int_{T_B} \frac{\rho(z)^{r_1 q_1 + a_1\delta_1 q_1 + \beta_1}}{|\rho(z,u)|^{c_1\delta_1 q_1}} dV(z) \lesssim \rho(u)^{s_1 q_1 - (b_1-\alpha_1)\delta_1 q_1},$$

which, together with (5.8), (5.9), and Lemma 3.1, we get

$$\begin{aligned}
&\int_{T_B}\left[\int_{T_B} [K_1(z,u)]^{\delta_1 q_1}\,[K_2(w,\eta)]^{\delta_2 q_1}\,[h_2(z,w)]^{q_1}\,dV_{\beta_1}(z)\right]^{q_2/q_1} dV_{\beta_2}(w)\\
&\lesssim \rho(u)^{s_1 q_2}\rho(\eta)^{(b_2-\alpha_2)\delta_2 q_2}\int_{T_B}\frac{\rho(w)^{r_2 q_2 + a_2\delta_2 q_2 + \beta_2}}{|\rho(w,\eta)|^{c_2\delta_2 q_2}} dV(w)\\
&\lesssim \rho(u)^{s_1 q_2}\rho(\eta)^{s_2 q_2} = [h_1(u,\eta)]^{q_2}.
\end{aligned}$$

Thus, condition (3.2) holds ture for the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$.

Therefore, the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ satisfies all the conditions of Lemma 3.5, then operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$. □

Lemma 5.2. *Let $1 = p_+ \le q_- \le q_+ < \infty$. If the parameters satisfy for any $i \in \{1,2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1,\ \ \alpha_i < b_i, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n+1+\beta_i}{q_i} - n - 1 - \alpha_i.$$

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.

Proof. When $p_1 = p_2 = 1$, for any $i \in \{1,2\}$, write $\gamma_i = \frac{s_i - r_i}{\tau_i}$, $\delta_i = \frac{(n+1+\beta_i)/q_i + r_i - s_i}{\tau_i}$. According to the proof of Lemma 5.1, we have

$$-\frac{\beta_i + 1}{q_i} - a_i\delta_i < r_i < a_i\gamma_i \tag{5.10}$$

and

$$-(b_i - \alpha_i)\,\gamma_i < s_i < (b_i - \alpha_i)\,\delta_i. \tag{5.11}$$

Next, we use Lemma 3.6 to prove this lemma.

First, we consider

$$\begin{aligned}
&(K_1(z,u))^{\gamma_1}\,(K_2(w,\eta))^{\gamma_2}\,h_1(u,\eta)\\
&= \frac{\rho(z)^{a_1\gamma_1}\rho(w)^{a_2\gamma_2}\rho(u)^{\gamma_1(b_1-\alpha_1)+s_1}\,\rho(\eta)^{\gamma_2(b_2-\alpha_2)+s_2}}{|\rho(z,u)|^{c_1\gamma_1}\,|\rho(w,\eta)|^{c_2\gamma_2}}.
\end{aligned}$$

By lemma 3.2, we have for any $z \in T_B$ and $u \in T_B$,

$$2\left|\rho\left(z,u\right)\right| \geq \max\left\{\rho\left(z\right),\rho\left(u\right)\right\}.$$

For any $i \in \{1,2\}$, we get

$$c_i\gamma_i = \left(b_i - \alpha_i + a_i\right)\gamma_i + s_i - r_i \tag{5.12}$$

due to $\tau_i = c_i - a_i - b_i + \alpha_i$ and $\gamma_i = \left(s_i - r_i\right)/\tau_i$.

From the first inequality in (5.11), we have

$$\gamma_i\left(b_i - \alpha_i\right) + s_i > 0. \tag{5.13}$$

Then, according to Lemma 3.2 and (5.12), for any given $z \in T_B$ and $u \in T_B$, we have

$$\begin{aligned}\frac{\rho\left(z\right)^{a_1\gamma_1}\rho\left(u\right)^{\left(b_1-\alpha_1\right)\gamma_1+s_1}}{\left|\rho\left(z,u\right)\right|^{c_1\gamma_1}} &= \left(\frac{\rho\left(u\right)}{\left|\rho\left(z,u\right)\right|}\right)^{\left(b_1-\alpha_1\right)\gamma_1+s_1}\left(\frac{\rho\left(z\right)}{\left|\rho\left(z,u\right)\right|}\right)^{a_1\gamma_1}\\ &\lesssim \rho(z)^{r_1}\end{aligned}$$

and, similarly, for any given $w \in T_B$ and any $\eta \in T_B$,

$$\begin{aligned}\frac{\rho\left(w\right)^{a_2\gamma_2}\rho\left(\eta\right)^{\left(b_2-\alpha_2\right)\gamma_2+s_2}}{\left|\rho\left(w,\eta\right)\right|^{c_2\gamma_2}} &= \left(\frac{\rho\left(\eta\right)}{\left|\rho\left(w,\eta\right)\right|}\right)^{\left(b_2-\alpha_2\right)\gamma_2+s_2}\left(\frac{\rho\left(w\right)}{\left|\rho\left(w,\eta\right)\right|}\right)^{a_2\gamma_2}\\ &\lesssim \rho(w)^{r_2}.\end{aligned}$$

Thus, for any given $(z,w) \in \mathcal{D}$,

$$\operatorname*{ess\,sup}_{(u,\eta)\in\mathcal{D}}\left(K_1\left(z,u\right)\right)^{\gamma_1}\left(K_2\left(w,\eta\right)\right)^{\gamma_2}h_1\left(u,\eta\right) \leq Ch_2\left(z,w\right).$$

Thus, condition (3.3) holds true for the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$.

Note that the condition (3.4) in Lemma 3.6 is the same as the condition (3.2) in Lemma 3.5, thus according to the proof of the second part of Lemma 5.1, we know that the condition (3.4) in Lemma 3.6 still holds.

Therefore, the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ satisfies all the conditions of Lemma 3.6, then operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}\left(\mathcal{D}\right)$ to $L^{\vec{q}}_{\vec{\beta}}\left(\mathcal{D}\right)$. □

Lemma 5.3. *Let $1 = p_2 < p_1 \leq q_- \leq q_+ < \infty$. If the parameters satisfy that, for any $i \in \{1,2\}$,*

$$\begin{cases}\alpha_1 + 1 < p_1\left(b_1+1\right),\ \alpha_2 < b_2,\\ -q_ia_i < \beta_i + 1,\ c_i = n+1+a_i+b_i+\lambda_i,\end{cases}$$

where

$$\begin{cases}\lambda_1 = \frac{n+1+\beta_1}{q_1} - \frac{n+1+\alpha_1}{p_1},\\ \lambda_2 = \frac{n+1+\beta_2}{q_2} - \left(n+1+\alpha_2\right).\end{cases}$$

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}\left(\mathcal{D}\right)$ to $L^{\vec{q}}_{\vec{\beta}}\left(\mathcal{D}\right)$.

Proof. For any $i \in \{1,2\}$, let $\tau_i = c_i - a_i - b_i + \alpha_i$, then

$$\tau_1 = \frac{n+1+\alpha_1}{p_1'} + \frac{n+1+\beta_1}{q_1} > 0$$

and

$$\tau_2 = \frac{n+1+\beta_2}{q_2} > 0.$$

Since $-q_1 a_1 < \beta_1 + 1$, we have

$$-\frac{\tau_1(\beta_1+1)}{q_1} < a_1\tau_1,$$

which is equivalent to

$$-\frac{\tau_1(\beta_1+1)}{q_1} - \frac{a_1(n+1+\beta_1)}{q_1} < \frac{a_1(n+1+\alpha_1)}{p_1'}. \tag{5.14}$$

In addition, from $\alpha_1 + 1 < p_1(b_1+1)$, it follows that

$$b_1 - \alpha_1 + \frac{\alpha_1+1}{p_1'} > 0.$$

Combined with the known fact that $b_2 > \alpha_2$, further implies that

$$-\frac{\tau_1(1+\alpha_1)}{p_1'} - \frac{(b_1-\alpha_1)(n+1+\alpha_1)}{p_1'} < \frac{(b_1-\alpha_1)(n+1+\beta_1)}{q_1}. \tag{5.15}$$

Thus, by (5.14) and (5.15), there exists r_1 and s_1 such that

$$-\frac{\tau_1(\beta_1+1)}{q_1} - \frac{a_1(n+1+\beta_1)}{q_1} < r_1\tau_1 + a_1(r_1 - s_1) < \frac{a_1(n+1+\alpha_1)}{p_1'}$$

and

$$\begin{cases} \frac{\tau_1(1+\alpha_1)}{p_1'} - \frac{(b_1-\alpha_1)(n+1+\alpha_1)}{p_1'} < \tau_1 s_1 + (b_1-\alpha_1)(s_1 - r_1) \\ \tau_1 s_1 + (b_1-\alpha_1)(s_1-r_1) < \frac{(b_1-\alpha_1)(n+1+\beta_1)}{q_1} \end{cases}.$$

Since $-q_2 a_2 < \beta_2 + 1$, we have

$$-\frac{\tau_2(\beta_2+1)}{q_2} < a_2\tau_2,$$

which is equivalent to

$$-\frac{\tau_2(\beta_2+1)}{q_2} - \frac{a_2(n+1+\beta_2)}{q_2} < 0. \tag{5.16}$$

In addition, from $\alpha_2 < b_2$, it follows that

$$\frac{(b_2-\alpha_2)(n+1+\beta_2)}{q_2} > 0. \tag{5.17}$$

Thus, by (5.16) and (5.17), there exists r_2 and s_2 such that

$$-\frac{\tau_2(\beta_2+1)}{q_2} - \frac{a_2(n+1+\beta_2)}{q_2} < r_2\tau_2 + a_2(r_2 - s_2) < 0$$

and

$$0 < \tau_2 s_2 + (b_2-\alpha_2)(s_2-r_2) < \frac{(b_2-\alpha_2)(n+1+\beta_2)}{q_2}.$$

That is

$$-\frac{\beta_i+1}{q_i}-a_i\delta_i<r_i<a_i\gamma_i, \tag{5.18}$$

$$-\frac{1+\alpha_1}{p_1'}-(b_1-\alpha_1)\gamma_1<s_1<(b_1-\alpha_1)\delta_1 \tag{5.19}$$

and

$$-(b_2-\alpha_2)\gamma_2<s_2<(b_2-\alpha_2)\delta_2, \tag{5.20}$$

where

$$\gamma_1=\frac{(n+1+\alpha_1)/p_1'+s_1-r_1}{\tau_1},$$

$$\delta_1=\frac{(n+1+\beta_1)/q_1+r_1-s_1}{\tau_1},$$

$$\gamma_2=\frac{s_2-r_2}{\tau_2}$$

and

$$\delta_2=\frac{(n+1+\beta_2)/q_2+r_2-s_2}{\tau_2}.$$

Clearly, $\gamma_1+\delta_1=1$, $\gamma_2+\delta_2=1$.

Now, let $h_1(u,\eta)=\rho(u)^{s_1}\rho(\eta)^{s_2}$, $h_2(z,w)=\rho(z)^{r_1}\rho(w)^{r_2}$, write $S_{\vec{a},\,\vec{b},\,\vec{c}}$ as

$$S_{\vec{a},\,\vec{b},\,\vec{c}}\,f(z,w)=\int_{T_B}\int_{T_B}K_1(z,u)\,K_2(w,\eta)f(u,\eta)\,dV_{\alpha_1}(u)\,dV_{\alpha_2}(\eta),$$

where

$$K_1(z,u)=\frac{\rho(u)^{b_1-\alpha_1}\rho(z)^{a_1}}{|\rho(z,u)|^{c_1}}\text{ and }K_2(w,\eta)=\frac{\rho(\eta)^{b_2-\alpha_2}\rho(w)^{a_2}}{|\rho(w,\eta)|^{c_2}}.$$

Next we prove this result by Lemma 3.7.

We consider

$$\begin{aligned}&\int_{T_B}(K_1(z,u))^{\gamma_1p_1'}(K_2(w,\eta))^{\gamma_2p_1'}(h_1(u,\eta))^{p_1'}\,dV_{\alpha_1}(u)\\&=\frac{\rho(z)^{a_1\gamma_1p_1'}\rho(w)^{a_2\gamma_2p_2'}\rho(\eta)^{(b_2-\alpha_2)\gamma_2p_1'+s_2p_1'}}{|\rho(w,\eta)|^{c_2\gamma_2p_1'}}\\&\quad\times\int_{T_B}\frac{\rho(u)^{(b_1-\alpha_1)\gamma_1p_1'+s_1p_1'+\alpha_1}}{|\rho(z,u)|^{c_1\gamma_1p_1'}}dV(u).\end{aligned} \tag{5.21}$$

From the first inequality in (5.19), we have

$$(b_1-\alpha_1)\gamma_1p_1'+s_1p_1'+\alpha_1>-1. \tag{5.22}$$

Notice that

$$(c_1-a_1-b_1+\alpha_1)\gamma_1=\tau_1\gamma_1=\frac{n+1+\alpha_1}{p_1'}+s_1-r_1,$$

by the second inequality in (5.18), we have

$$c_1\gamma_1p_1'-n-1-(b_1-\alpha_1)\gamma_1p_1'-s_1p_1'-\alpha_1=a_1\gamma_1p_1'-r_1p_1'>0. \tag{5.23}$$

In addition, from Lemma 3.2 and the fact $c_2\gamma_2+r_2-a_2\gamma_2=(b_2-\alpha_2)\gamma_2+s_2>0$, we infer that, for any $w\in T_B$ and $\eta\in T_B$,

$$\frac{\rho(\eta)^{(b_2-\alpha_2)\gamma_2p_1'+s_2p_1'}}{\rho(w,\eta)^{c_2\gamma_2p_1'}}\lesssim\rho(w)^{(r_2-a_2\gamma_2)p_1'}. \tag{5.24}$$

Thus, according to Lemma 3.1, for any given $z\in T_B$, we have

$$\int_{T_B}\frac{\rho(u)^{(b_1-\alpha_1)\gamma_1p_1'+s_1p_1'+\alpha_1}}{|\rho(z,u)|^{c_1\gamma_1p_1'}}dV(u)\lesssim\rho(z)^{r_1p_1'-a_1\gamma_1p_1'}, \tag{5.25}$$

combining with (5.24) and (5.25), we further obtain that

$$\begin{aligned}&\operatorname*{ess\,sup}_{\eta\in T_B}\int_{T_B}(K_1(z,u))^{\gamma_1p_1'}(K_2(w,\eta))^{\gamma_2p_1'}(h_1(u,\eta))^{p_1'}dV_{\alpha_1}(u)\\&\lesssim\rho(z)^{r_1p_1'}\rho(w)^{r_2p_1'}=(h_2(z,w))^{p_1'}.\end{aligned}$$

Thus, condition (3.5) holds true for the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$.

Next, we verify condition (3.6) of Lemma 3.7. Notice that

$$\begin{aligned}&\int_{T_B}(K_1(z,u))^{\delta_1q_1}(K_2(w,\eta))^{\delta_2q_1}(h_2(z,w))^{q_1}dV_{\beta_1}(z)\\&=\frac{\rho(u)^{(b_1-\alpha_1)\delta_1q_1}\rho(\eta)^{(b_2-\alpha_2)\delta_2q_1}\rho(w)^{r_2q_1+a_2\delta_2q_1}}{|\rho(w,\eta)|^{c_2\delta_2q_1}}\\&\quad\times\int_{T_B}\frac{\rho(z)^{r_1q_1+a_1\delta_1q_1+\beta_1}}{|\rho(z,u)|^{c_1\delta_1q_1}}dV(z).\end{aligned}$$

From the first inequality in (5.18), we obviously have

$$r_iq_i+a_i\delta_iq_i+\beta_i>-1. \tag{5.26}$$

Notice that

$$(c_i-a_i-b_i+\alpha_i)\delta_i=\tau_i\delta_i=\frac{n+1+\beta_i}{q_i}+r_i-s_i,$$

by the second inequality in (5.19) and (5.20), we have

$$c_i\delta_iq_i-n-1-a_i\delta_iq_i-r_iq_i-\beta_i=(b_i-\alpha_i)\delta_iq_i-s_iq_i>0. \tag{5.27}$$

Thus, according to Lemma 3.1, for any given $u\in T_B$, we have

$$\int_{T_B}\frac{\rho(z)^{r_1q_1+a_1\delta_1q_1+\beta_1}}{|\rho(z,u)|^{c_1\delta_1q_1}}dV(u)\lesssim\rho(u)^{s_1q_1-(b_1-\alpha_1)\delta_1q_1}, \tag{5.28}$$

combining with (5.26), (5.27) and Lemma 3.1, this imply that

$$\begin{aligned}&\int_{T_B}\left(\int_{T_B}(K_1(z,u))^{\delta_1q_1}(K_2(w,\eta))^{\delta_2q_1}(h_2(z,w))^{q_1}dV_{\beta_1}(z)\right)^{q_2/q_1}dV_{\beta_2}(w)\\&\lesssim\rho(\eta)^{(b_2-\alpha_2)\delta_2q_2}\rho(u)^{s_1q_2}\int_{T_B}\frac{\rho(w)^{r_2q_2+a_2\delta_2q_2+\beta_2}}{|\rho(w,\eta)|^{c_2\delta_2q_2}}dV(w)\\&\lesssim\rho(u)^{s_1q_2}\rho(\eta)^{s_2q_2}.\end{aligned}$$

Thus, condition (3.6) holds ture for the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$.

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ satisfies all the conditions of Lemma 3.7, therefore, operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$. □

Lemma 5.4. *Let $1 = p_1 < p_2 \le q_- \le q_+ < \infty$. If the parameters satisfy that, for any $i \in \{1,2\}$,*

$$\begin{cases} \alpha_1 < b_1,\ (\alpha_2+1) < p_2(b_2+1), \\ -q_i a_i < \beta_i + 1,\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = \frac{n+1+\beta_1}{q_1} - (n+1+\alpha_1), \\ \lambda_2 = \frac{n+1+\beta_2}{q_2} - \frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.

Proof. Lemma 5.4 is the symmetric case of Lemma 5.3. We only need to modify the definitions of γ_1, δ_1, γ_2 and δ_2, thus we omit the proof here. □

Lemma 5.5. *Let $1 < p_1 < q_1 = \infty$ and $1 < p_2 < q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1,2\}$,*

$$\begin{cases} a_i > 0,\ \alpha_i + 1 < p_i(b_i+1), \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\lambda_i = -\frac{n+1+\alpha_i}{p_i}.$$

Then the operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. According to the definition of operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$, we know that the kernel of the integral is

$$K_1(z,u)\cdot K_2(w,\eta) = \frac{\rho(z)^{a_1}\rho(u)^{b_1-\alpha_1}\rho(w)^{a_2}\rho(\eta)^{b_2-\alpha_2}}{|\rho(z,u)|^{c_1}\,|\rho(w,\eta)|^{c_2}}.$$

By Lemma 3.9, it suffices to prove

$$\operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}} \|K_1(z,\cdot)\cdot K_2(w,\cdot)\|_{L^{\vec{p}'}_{\vec{\alpha}}} < \infty.$$

By calculation, we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}} \|K_1(z,\cdot)\cdot K_2(w,\cdot)\|_{L^{\vec{p}'}_{\vec{\mu}}} \\ &\quad= \operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}} \rho(z)^{a_1}\left(\int_{T_B}\frac{\rho(u)^{p_1'(b_1-\alpha_1)+\alpha_1}}{|\rho(z,u)|^{c_1p_1'}}dV(u)\right)^{\frac{1}{p_1'}} \\ &\qquad\times \rho(w)^{a_2}\left(\int_{T_B}\frac{\rho(\eta)^{p_2'(b_2-\alpha_2)+\alpha_2}}{|\rho(w,\eta)|^{c_2p_2'}}dV(\eta)\right)^{\frac{1}{p_2'}}. \end{aligned}$$

Combining the known conditions and Lemma 3.1, we can easily conclude that

$$\left\|K_1(z,\cdot)\cdot K_2(w,\cdot)\right\|_{L^{\vec{p}'}_{\vec{\alpha}}}$$

is uniformly bounded with respect to z and w.

Therefore, the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$. □

Lemma 5.6. *Let $1=p_1<q_1=\infty$ and $1=p_2<q_2=\infty$. If the parameters satisfy that, for any $i\in\{1,2\}$,*

$$\begin{cases} a_i\geq 0,\ b_i\geq \alpha_i,\\ c_i=n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\lambda_i=-n-1-\alpha_i.$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. By Lemma 3.9, it suffices to prove

$$\operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}}\left\|K_1(z,\cdot)\cdot K_2(w,\cdot)\right\|_{\infty}<\infty.$$

By calculation, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}}\left\|K_1(z,\cdot)\cdot K_2(w,\cdot)\right\|_{\infty}\\ &\quad=\operatorname*{ess\,sup}_{(z,w)\in\mathcal{D}}\operatorname*{ess\,sup}_{(u,\eta)\in\mathcal{D}}\rho(z)^{a_1}\frac{\rho(u)^{b_1-\alpha_1}}{|\rho(z,u)|^{c_1}}\rho(w)^{a_2}\frac{\rho(\eta)^{b_2-\alpha_2}}{|\rho(w,\eta)|^{c_2}}.\end{aligned}$$

Combined with the known conditions, by lemma 3.2, we can easily conclude that

$$\left\|K_1(z,\cdot)\cdot K_2(w,\cdot)\right\|_{\infty}$$

is uniformly bounded with respect to z and w.

Therefore, the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$. □

Lemma 5.7. *Let $1=p_1<q_1=\infty$ and $1<p_2<q_2=\infty$. If the parameters satisfy that, for any $i\in\{1,2\}$,*

$$\begin{cases} a_1\geq 0,\ a_2>0,\\ b_1\geq \alpha_1,\ p_2(b_2+1)>\alpha_2+1,\\ c_i=n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases}\lambda_1=-n-1-\alpha_1,\\ \lambda_2=-\frac{n+1+\alpha_2}{p_2}.\end{cases}$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. This lemma is a direct corollary of Lemma 5.5, Lemma 5.6 and Lemma 3.9, which we omit to prove. □

Lemma 5.8. *Let $1 < p_1 < q_1 = \infty$ and $1 = p_2 < q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 > 0,\ a_2 \geq 0, \\ p_1 (b_1 + 1) > \alpha_1 + 1,\ b_2 \geq \alpha_2, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -\frac{n+1+\alpha_1}{p_1}, \\ \lambda_2 = -n - 1 - \alpha_2. \end{cases}$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. Lemma 5.8 is the symmetric case of Lemma 5.7. Thus, we omit the proof here. □

Lemma 5.9. *Let $p_1 = q_1 = \infty$ and $p_2 = q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_i > 0,\ b_i > -1, \\ c_i = n + 1 + a_i + b_i. \end{cases} \tag{5.29}$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. Obviously, when the parameter satisfies condition (5.29),

$$\begin{aligned} &\left| S_{\vec{a},\ \vec{b},\ \vec{c}}\, f(z, w) \right| \\ &\lesssim \rho(z)^{a_1} \int_{T_B} \frac{\rho(u)^{b_1}}{|\rho(z, u)|^{c_1}} dV(u) \cdot \rho(w)^{a_2} \int_{T_B} \frac{\rho(\eta)^{b_2}}{|\rho(w, \eta)|^{c_2}} dV(\eta) \\ &< \infty. \end{aligned}$$

Thus, by Lemma 3.1, $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$. □

Lemma 5.10. *Let $1 = p_1 < q_1 = \infty$ and $p_2 = q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 \geq 0,\ a_2 > 0, \\ b_1 \geq \alpha_1,\ b_2 > -1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -n - 1 - \alpha_1, \\ \lambda_2 = 0. \end{cases}$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. The proof of this lemma is similar to Lemma 4.10. Thus, we omit the proof. □

Lemma 5.11. *Let $p_1 = q_1 = \infty$ and $1 = p_2 < q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 > 0, \ a_2 \geq 0, \\ b_1 > -1, \ b_2 \geq \alpha_2, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -n - 1 - \alpha_2. \end{cases}$$

Then $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. Lemma 5.10 is the symmetric case of Lemma 5.9. Thus, we omit the proof here. □

Lemma 5.12. *Let $p_1 = q_1 = \infty$ and $1 < p_2 < q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 > 0, \ a_2 > 0, \\ b_1 > -1, \ p_2 (b_2 + 1) > \alpha_2 + 1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -\frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Then the operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. The proof of this lemma is similar to Lemma 4.12. Thus, we omit the proof. □

Lemma 5.13. *Let $1 < p_1 < q_1 = \infty$ and $p_2 = q_2 = \infty$. If the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 > 0, \ a_2 > 0, \\ p_1 (b_1 + 1) > \alpha_1 + 1, \ b_2 > -1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -\frac{n+1+\alpha_2}{p_2}, \\ \lambda_2 = 0. \end{cases}$$

Then $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.

Proof. Lemma 5.13 is the symmetric case of Lemma 5.12. Thus, we omit the proof here. □

6. The Proof of Main Theorems

In this section, we will list all the main theorems of this paper and prove them. Here are 13 main theorems. We give an explanation of the proof of the first theorem. The other theorems correspond to the lemmas in Sects. 3 and 4 one by one. In order to be concise, we will not repeat them.

Theorem 6.1. *Let $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_- \le p_+ \le q_- \le q_+ < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1,\ \alpha_i + 1 < p_i\,(b_i + 1)\,, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n + 1 + \beta_i}{q_i} - \frac{n + 1 + \alpha_i}{p_i}.$$

Proof. (1) $\Rightarrow$ (2) is trivial; (2) $\Rightarrow$ (3) comes from Lemma 4.1; (3) $\Rightarrow$ (1) is derived from Lemma 5.1. □

Theorem 6.2. *Let $\vec{p} := (1, 1)$ and $\vec{q} := (q_1, q_2) \in [1, \infty) \times [1, \infty)$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} -q_i a_i < \beta_i + 1,\ \alpha_i < b_i, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n + 1 + \beta_i}{q_i} - n - 1 - \alpha_i.$$

Theorem 6.3. *Let $\vec{p} := (p_1, 1)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_1 \le q_- \le q_+ < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} \alpha_1 + 1 < p_1\,(b_1 + 1)\,,\ \alpha_2 < b_2, \\ -q_i a_i < \beta_i + 1,\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = \frac{n+1+\beta_1}{q_1} - \frac{n+1+\alpha_1}{p_1}, \\ \lambda_2 = \frac{n+1+\beta_2}{q_2} - (n + 1 + \alpha_2)\,. \end{cases}$$

Theorem 6.4. *Let $\vec{p} := (1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_2 \le q_- \le q_+ < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} \alpha_1 < b_1,\ (\alpha_2 + 1) < p_2(b_2 + 1), \\ -q_i a_i < \beta_i + 1,\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = \frac{n+1+\beta_1}{q_1} - (n + 1 + \alpha_1), \\ \lambda_2 = \frac{n+1+\beta_2}{q_2} - \frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Theorem 6.5. *Let $\vec{p} := (p_1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_- \le p_+ < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_i > 0,\ \alpha_i + 1 < p_i(b_i + 1), \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = -\frac{n + 1 + \alpha_i}{p_i}.$$

Theorem 6.6. *Let $\vec{p} := (1, 1)$ and $\vec{q} := (\infty, \infty)$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(3) The parameters satisfy that, for any $i \in \{1, 2\}$,

$$\begin{cases} a_i \ge 0,\ b_i \ge \alpha_i, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = -n - 1 - \alpha_i.$$

Theorem 6.7. *Let $\vec{p} := (1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_2 < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(2) *The operator $T_{\vec{a},\,\vec{b},\,\vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} a_1 \ge 0,\ a_2 > 0, \\ b_1 \ge \alpha_1,\ p_2\,(b_2 + 1) > \alpha_2 + 1, \\ c_i = n + 1 + a_i + b_i + \lambda_i, \end{cases}$$

where

$$\begin{cases}\lambda_1 = -n-1-\alpha_1,\\ \lambda_2 = -\frac{n+1+\alpha_2}{p_2}.\end{cases}$$

Theorem 6.8. *Let $\vec{p} := (p_1, 1)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_1 < \infty$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases}a_1 > 0,\ a_2 \geq 0,\\ p_1(b_1+1) > \alpha_1 + 1,\ b_2 \geq \alpha_2,\\ c_i = n+1+a_i+b_i+\lambda_i,\end{cases}$$

where

$$\begin{cases}\lambda_1 = -\frac{n+1+\alpha_1}{p_1},\\ \lambda_2 = -n-1-\alpha_2.\end{cases}$$

Theorem 6.9. *Let $\vec{p} := (\infty, \infty)$ and $\vec{q} := (\infty, \infty)$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases}a_i > 0,\ b_i > -1,\\ c_i = n+1+a_i+b_i.\end{cases}$$

Theorem 6.10. *Let $\vec{p} := (1, \infty)$ and $\vec{q} := (\infty, \infty)$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(3) *The parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases}a_1 \geq 0,\ a_2 > 0,\\ b_1 \geq \alpha_1,\ b_2 > -1,\\ c_i = n+1+a_i+b_i+\lambda_i,\end{cases}$$

where

$$\begin{cases}\lambda_1 = -n-1-\alpha_1,\\ \lambda_2 = 0.\end{cases}$$

Theorem 6.11. *Let $\vec{p} := (\infty, 1)$ and $\vec{q} := (\infty, \infty)$. Then the following conclusions are equivalent.*

(1) *The operator $S_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*
(2) *The operator $T_{\vec{a},\ \vec{b},\ \vec{c}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

(3) *The parameters satisfy that, for any* $i \in \{1,2\}$,

$$\begin{cases} a_1 > 0,\ a_2 \geq 0, \\ b_1 > -1,\ b_2 \geq \alpha_2, \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -n-1-\alpha_2. \end{cases}$$

Theorem 6.12. *Let* $\vec{p} := (\infty, p_2)$ *and* $\vec{q} := (\infty,\infty)$ *satisfy* $1 < p_2 < \infty$. *Then the following conclusions are equivalent.*

(1) *The operator* $S_{\vec{a},\ \vec{b},\ \vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\infty}(\mathcal{D})$.
(2) *The operator* $T_{\vec{a},\ \vec{b},\ \vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\infty}(\mathcal{D})$.
(3) *The parameters satisfy that, for any* $i \in \{1,2\}$,

$$\begin{cases} a_1 > 0,\ a_2 > 0, \\ b_1 > -1,\ p_2(b_2+1) > \alpha_2+1, \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -\frac{n+1+\alpha_2}{p_2}. \end{cases}$$

Theorem 6.13. *Let* $\vec{p} := (p_1, \infty)$ *and* $\vec{q} := (\infty,\infty)$ *satisfy* $1 < p_1 < \infty$. *Then the following conclusions are equivalent.*

(1) *The operator* $S_{\vec{a},\ \vec{b},\ \vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\infty}(\mathcal{D})$.
(2) *The operator* $T_{\vec{a},\ \vec{b},\ \vec{c}}$ *is bounded from* $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ *to* $L^{\infty}(\mathcal{D})$.
(3) *The parameters satisfy that, for any* $i \in \{1,2\}$,

$$\begin{cases} a_1 > 0,\ a_2 > 0, \\ p_1(b_1+1) > \alpha_1+1,\ b_2 > -1, \\ c_i = n+1+a_i+b_i+\lambda_i, \end{cases}$$

where

$$\begin{cases} \lambda_1 = -\frac{n+1+\alpha_1}{p_1}, \\ \lambda_2 = 0. \end{cases}$$

7. Applications

In this section, we will give three applications of the main theorems in this paper, that is, to study the boundedness of three kinds of special integral operators.

We study the $L^{\vec{p}}_{\vec{\gamma}}(\mathcal{D})$-$L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ boundedness of the following operators

$$T^{\vec{\gamma}}_{\vec{c}} f(z,w) = \int_{T_B}\int_{T_B} \frac{f(u,\eta)}{\rho(z,u)^{c_1}\,\rho(w,\eta)^{c_2}} dV_{\gamma_1}(u) dV_{\gamma_2}(\eta),$$

where $c_1,\ c_2 > 0$ and $\gamma_1,\ \gamma_2 > -1$.

Notice that $T_{\vec{c}}^{\vec{\gamma}} = T_{\vec{0},\ \vec{\gamma},\ \vec{c}}$, we have the following results.

Corollaries 7.1 and 7.2 state when operator $T_{\vec{c}}^{\vec{\gamma}}$ is bounded.

Corollary 7.1. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_- \le p_+ \le q_- \le q_+ < \infty$, then the operator $T_{\vec{c}}^{\vec{\gamma}}$ is bounded from $L_{\vec{\gamma}}^{\vec{p}}(\mathcal{D})$ to $L_{\vec{\beta}}^{\vec{q}}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} \beta_i > -1, \\ c_i = n + 1 + \gamma_i + \lambda_i, \end{cases}$$

where

$$\lambda_i = \frac{n + 1 + \beta_i}{q_i} - \frac{n + 1 + \gamma_i}{p_i}.$$

Corollary 7.2. *If $\vec{p} := (1, 1)$ and $\vec{q} := (\infty, \infty)$, then the operator $T_{\vec{c}}^{\vec{\gamma}}$ is bounded from $L_{\vec{\gamma}}^{\vec{1}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if $c_1 = c_2 = 0$.*

Corollary 7.3 states when operator $T_{\vec{c}}^{\vec{\gamma}}$ is unbounded.

Corollary 7.3. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy one of the following conditions :*

(1) *$p_1 = p_2 = 1$ and $q_1, q_2 \ge 1$;*
(2) *$1 = p_- < p_+ < q_- \le q_+ < \infty$;*
(3) *$1 \le p_- \le p_+ \le \infty$, $\vec{p} \ne (1, 1)$ and $(q_1, q_2) = (\infty, \infty)$; then the operator $T_{\vec{c}}^{\vec{\gamma}}$ is unbounded on $L_{\vec{\gamma}}^{\vec{p}}(\mathcal{D})$.*

Next, we study the boundedness of Bergman-type projections.

Let $\vec{\gamma} := (\gamma_1, \gamma_2) \in (-1, \infty)^2$, the weighted multiparameter Bergman-type projection is as follows :

$$P_{\vec{\gamma}} f(z, w) := \int_{T_B} \int_{T_B} \frac{f(u, \eta)}{\rho(z, u)^{n+1+\gamma_1} \rho(w, \eta)^{n+1+\gamma_2}} dV_{\gamma_1}(u) dV_{\gamma_2}(\eta).$$

Notice that $P_{\vec{\gamma}} = T_{\overrightarrow{n+1+\gamma}}^{\vec{\gamma}} = T_{\vec{0},\ \vec{\gamma},\ \overrightarrow{n+1+\gamma}}$, hence we have the following results.

The following corollaries 7.4 to 7.8 give the case when the operator $P_{\vec{\gamma}}$ is bounded from $L_{\vec{\alpha}}^{\vec{p}}(\mathcal{D})$ to $L_{\vec{\beta}}^{\vec{q}}(\mathcal{D})$. Corollary 7.9 gives when the operator $P_{\vec{\gamma}}$ is unbounded on $L_{\vec{\alpha}}^{\vec{p}}(\mathcal{D})$.

Corollary 7.4. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_- \le p_+ \le q_- \le q_+ < \infty$, then the operator $P_{\vec{\gamma}}$ is bounded from $L_{\vec{\alpha}}^{\vec{p}}(\mathcal{D})$ to $L_{\vec{\beta}}^{\vec{q}}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} p_i(\gamma_i + 1) > \alpha_i + 1, \\ p_i(n + 1 + \beta_i) = q_i(n + 1 + \alpha_i). \end{cases}$$

Corollary 7.5. *If $\vec{p} := (1,1)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 \leq q_- \leq q_+ < \infty$, then the operator $P_{\vec{\gamma}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1,2\}$,*

$$\begin{cases} \gamma_i > \alpha_i, \\ n+1+\beta_i = q_i\,(n+1+\alpha_i)\,. \end{cases}$$

Corollary 7.6. *If $\vec{p} := (p_1, 1)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_1 \leq q_- \leq q_+ < \infty$, then the operator $P_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} p_1\,(\gamma_1+1) > \alpha_1+1,\ \gamma_2 > \alpha_2, \\ p_1\,(n+1+\beta_1) = q_1\,(n+1+\alpha_1)\,, \\ n+1+\beta_2 = q_2\,(n+1+\alpha_2)\,. \end{cases}$$

Corollary 7.7. *If $\vec{p} := (1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_2 \leq q_- \leq q_+ < \infty$, then the operator $P_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} \gamma_1 > \alpha_1,\ p_2\,(\gamma_2+1) > \alpha_2+1, \\ n+1+\beta_1 = q_1\,(n+1+\alpha_1)\,, \\ p_2\,(n+1+\beta_2) = q_2\,(n+1+\alpha_2)\,. \end{cases}$$

Corollary 7.8. *If $\vec{p} := (1,1)$ and $\vec{q} := (\infty, \infty)$, then the operator $P_{\vec{\gamma}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1,2\}$,*

$$\begin{cases} \gamma_i \geq \alpha_i, \\ \alpha_i = -\,(n+1)\,. \end{cases}$$

Corollary 7.9. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy the following conditions :*

$$1 \leq p_- \leq p_+ \leq \infty,\ \vec{p} \neq (1,1) \ \text{ and } \ (q_1, q_2) = (\infty, \infty)\,,$$

then the operator $P_{\vec{\gamma}}$ is unbounded on $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$.

At the end of this section, we study the boundedness of the weighted multiparameter Berezin-type transform $B_{\vec{\gamma}}$ which is given by

$$B_{\vec{\gamma}} f := \int_{T_B} \int_{T_B} \frac{\rho\,(z)^{n+1+\gamma_1}\,\rho\,(w)^{n+1+\gamma_2}}{|\rho\,(z,u)|^{2(n+1+\gamma_1)}\,|\rho\,(w,\eta)|^{2(n+1+\gamma_2)}} f\,(u,\eta)\,dV_{\gamma_1}\,(u) dV_{\gamma_2}\,(\eta),$$

where $\vec{\gamma} := (\gamma_1, \gamma_2) \in (-1, \infty) \times (-1, \infty)$.

Notice that $B_{\vec{\gamma}} = S_{\overrightarrow{n+1+\gamma}, \vec{\gamma}, \overrightarrow{2(n+1+\gamma)}}$, therefore we have the following results.

The following corollaries 7.10 to 7.20 give the case when the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$.

Corollary 7.10. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_- \le p_+ \le q_- \le q_+ < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} -q_i\,(n+1+\gamma_i) < \beta_i + 1,\ \alpha_i + 1 < p_i\,(\gamma_i + 1)\,, \\ p_i\,(n+1+\beta_i) = q_i\,(n+1+\alpha_i)\,. \end{cases}$$

Corollary 7.11. *If $\vec{p} := (1, 1)$ and $\vec{q} := (q_1, q_2) \in [1, \infty) \times [1, \infty)$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\begin{cases} -q_i\,(n+1+\gamma_i) < \beta_i + 1,\ \alpha_i < \gamma_i, \\ n+1+\beta_i = q_i\,(n+1+\alpha_i)\,. \end{cases}$$

Corollary 7.12. *If $\vec{p} := (p_1, 1)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_1 \le q_- \le q_+ < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} p_1\,(\gamma_1 + 1) > \alpha_1 + 1,\ \gamma_2 > \alpha_2, \\ p_1\,(n+1+\beta_1) = q_1\,(n+1+\alpha_1)\,, \\ n+1+\beta_2 = q_2\,(n+1+\alpha_2)\,. \end{cases}$$

Corollary 7.13. *If $\vec{p} := (1, p_2)$ and $\vec{q} := (q_1, q_2)$ satisfy $1 < p_2 \le q_- \le q_+ < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\vec{q}}_{\vec{\beta}}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} \gamma_1 > \alpha_1,\ p_2\,(\gamma_2 + 1) > \alpha_2 + 1, \\ n+1+\beta_1 = q_1\,(n+1+\alpha_1)\,, \\ p_2\,(n+1+\beta_2) = q_2\,(n+1+\alpha_2)\,. \end{cases}$$

Corollary 7.14. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_- \le p_+ < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\gamma_i > \alpha_i = -n - 1.$$

Corollary 7.15. *If $\vec{p} := (1, 1)$ and $\vec{q} := (\infty, \infty)$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{1}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$,*

$$\gamma_i \ge \alpha_i = -n - 1.$$

Corollary 7.16. *If $\vec{p} := (1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_2 < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} \gamma_1 \ge \alpha_1 = -n - 1, \\ \gamma_2 > \alpha_2 = -n - 1. \end{cases}$$

Corollary 7.17. *If $\vec{p} := (p_1, 1)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_1 < \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that,*

$$\begin{cases} \gamma_1 > \alpha_1 = -n - 1, \\ \gamma_2 \geq \alpha_2 = -n - 1. \end{cases}$$

Corollary 7.18. *If $\vec{p} := (\infty, \infty)$ and $\vec{q} := (\infty, \infty)$, then the operator $B_{\vec{\gamma}}$ is always bounded from $L^{\infty}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$.*

Corollary 7.19. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 = p_- < p_+ = \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$, $\gamma_i \geq \alpha_i = -n - 1$.*

Corollary 7.20. *If $\vec{p} := (p_1, p_2)$ and $\vec{q} := (\infty, \infty)$ satisfy $1 < p_- < p_+ = \infty$, then the operator $B_{\vec{\gamma}}$ is bounded from $L^{\vec{p}}_{\vec{\alpha}}(\mathcal{D})$ to $L^{\infty}(\mathcal{D})$ if and only if the parameters satisfy that, for any $i \in \{1, 2\}$, $\gamma_i > \alpha_i = -n - 1$.*

Data availability No datasets were generated or analyzed during the current study.

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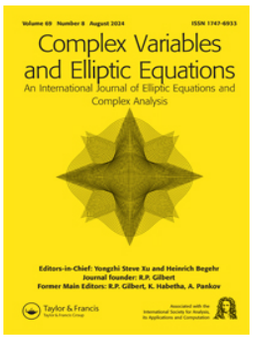
Received: August 27, 2024.
Accepted: May 28, 2025.



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Complex Variables and Elliptic Equations

An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: www.tandfonline.com/journals/gcov20

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To cite this article: Guangfu Cao & Haichou Li (2024) Composition operators on Dirichlet spaces over the half-plane, Complex Variables and Elliptic Equations, 69:8, 1365-1377, DOI: 10.1080/17476933.2023.2221638

To link to this article: https://doi.org/10.1080/17476933.2023.2221638

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COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
2024, VOL. 69, NO. 8, 1365–1377
https://doi.org/10.1080/17476933.2023.2221638

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Composition operators on Dirichlet spaces over the half-plane

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ABSTRACT

As a continuation of the study of polynomial approximation and composition operators on Dirichlet spaces of unit disk, which has settled a problem posed by Cima in 1976, the present paper aims to consider the case of the unbounded domains, such as the half-plane. Specifically, we may obtain the rational approximations in the Dirichlet spaces and characterize the composition operators which have a dense range on the Dirichlet spaces over the half-plane. Moreover, this paper also considers the relationship between the Dirichlet spaces and Hardy spaces on half-plane.

ARTICLE HISTORY
Received 20 October 2021
Accepted 22 May 2023

COMMUNICATED BY
H. Wulan

KEYWORDS
Dirichlet space; Hardy space; composition operator; rational approximation; univalent function

AMS SUBJECT CLASSIFICATIONS
47A05; 47A58; 46E50; 46E15

1. Introduction

As we all know, there have been a great number of studies on analytic composition operators acting on spaces of functions on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, such as [1–4] and so on. However, composition operators on the equivalent spaces over the upper half-plane $\mathbb{C}^+$ enjoyed considerably less attention, except [5–9]. The composition operator $C_\varphi : f \to f \circ \varphi$, where self-mapping $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic. Although many corresponding spaces of analytic functions in the unit disc and half-plane are isomorphic, composition operators act very differently in theses two settings. For example, the Littlewood principle shows that all composition operators acting on Hardy spaces of $\mathbb{D}$ are bounded, but Matache [7,8] proved that the Hardy spaces over $\mathbb{C}^+$ support no compact composition operators and not all are bounded. In addition, the Bergman spaces on $\mathbb{D}$ and $\mathbb{C}^+$ are also significantly different regarding the composition operators acting on them.

Based on the essential difference between the analytic composition operators acting on the spaces of functions on $\mathbb{D}$ and $\mathbb{C}^+$, this paper aims to consider the composition operators on the Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ over the upper half-plane.

In [10], J.A. Cima studied the Fredholmness of composition operators on the Dirichlet spaces $\mathfrak{D}(\mathbb{D})$, and raised the following problem.

Problem 1.1: *Characterize those φ in the unit ball of H^∞ for which C_φ has dense range.*

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Cima indicated that if C_φ has a dense range, then for $\psi(z) = r\varphi(z)$, $0 < r < 1$, C_ψ has a dense range. He said that no conjecture on this question. This is an old open problem relative to the density of polynomials in the Dirichlet spaces on the domain $\Omega = \varphi(\mathbb{D})$. The first author of this paper and his coworkers (K.H. Zhu and L. He) find that the density of the polynomials in $\mathfrak{D}(\Omega)$ is equivalent to the density of the range of the composition operators C_φ acting on $\mathfrak{D}(\mathbb{D})$, and obtain the following characterization results in [11], which partly solves the open Problem 1.1.

Theorem 1.1 ([11]): *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $\Omega = \varphi(\mathbb{D})$. Then C_φ is a bounded composition operator on Dirichlet spaces $\mathfrak{D}(\mathbb{D})$ with dense range if and only if the polynomials are dense in the Bergman space $A^2(\Omega)$ and the self-mapping φ is univalent.*

It is natural to ask how to solve this problem about the case of unbounded domain, more specifically, how to characterize the composition operators C_φ with dense range on the Dirichlet spaces over the upper half-plane. This is a nontrivial problem, because there indeed exist some essential difficulties, such as finding dense set in $\mathfrak{D}(\mathbb{C}^+)$ is not so easy. Unlike $\mathfrak{D}(\mathbb{D})$ where the polynomials are dense, dense sets in $\mathfrak{D}(\mathbb{C}^+)$ are often harder to come by. We first need some very smooth class of analytic functions that are dense in $\mathfrak{D}(\mathbb{C}^+)$ and will play the same role of the polynomials in the disc case $\mathfrak{D}(\mathbb{D})$, and then obtain the characterization of composition operators with dense range on $\mathfrak{D}(\mathbb{C}^+)$. Moreover, we also consider the boundedness of composition operators acting on $\mathfrak{D}(\mathbb{C}^+)$ and study the relationship between Hardy spaces $H^2(\mathbb{C}^+)$ and Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$.

The present paper is arranged as follows. The second section is about some preliminary knowledge on the Dirichlet spaces. The third section consider the density of rational functions in $\mathfrak{D}(\mathbb{C}^+)$. The fourth section characterizes the composition operators with dense range on $\mathfrak{D}(\mathbb{C}^+)$. The last section considers the relationship between Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ and Hardy spaces $H^2(\mathbb{C}^+)$ over the upper half-plane.

2. Preliminary knowledge

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{C}^+$ the upper half-plane of complex plane, i.e.

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \mathbf{Im}\, z > 0\} < \infty.$$

The Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ over the upper half-plane $\mathbb{C}^+$, denoted by $\mathfrak{D}(\mathbb{C}^+)$, consists of analytic functions f on $\mathbb{C}^+$ satisfying

$$\int_{\mathbb{C}^+} |f'(z)|^2 d_{A(z)}$$

is finite.

We here use $d_{A(z)}$ to denote Lebesgue (area) measure on $\mathbb{C}$ and its restriction to various domains, including $\mathbb{D}$ and $\mathbb{C}^+$. Moreover, we note that the norm of the function f in Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ is defined as

$$\|f\|^2_{\mathfrak{D}(\mathbb{C}^+)} = |f(i)|^2 + \int_{\mathbb{C}^+} |f'(z)|^2 d_{A(z)}.$$

It is easy to see that $\mathfrak{D}(\mathbb{C}^+)$ is a Hilbert space whose inner production is given by

$$\langle f, g\rangle = f(i)\overline{g(i)} + \int_{\mathbb{C}^+} f'(z)\overline{g'(z)}d_{A(z)}.$$

In order to state our main results, we still need to introduce the Dirichlet spaces and Bergman spaces of any domain in complex plane. For any domain Ω of upper half-plane C^+, the Dirichlet space of Ω, denoted by $\mathfrak{D}(\Omega)$, consists of all analytic functions f on Ω satisfying

$$\int_\Omega |f'(z)|^2 d_{A(z)}$$

is finite.

The Bergman spaces $A^2(\Omega)$ on Ω consist all analytic functions f on Ω satisfying

$$\int_\Omega |f(z)|^2 d_{A(z)} < \infty.$$

Let the fractional linear mapping

$$z = \alpha(w) = i\frac{1-w}{1+w} \tag{1}$$

be the conformal mapping of unit disc $\mathbb{D}$ on to the upper half plane $\mathbb{C}^+$.

This mapping has inverse

$$w = \alpha^{-1}(z) = \frac{i-z}{i+z}$$

which is from $\mathbb{C}^+$ to $\mathbb{D}$. We note that $\mathrm{d}\theta$ is the arc-length Lebesgue measure on the unit circle

$$\partial\mathbb{D} = \{w \in \mathbb{C} : |w| = 1\},$$

and $\mathrm{d}x$ is the Lebesgue measure on the real line $\mathbb{R}$.

We know that the fractional linear mapping $\alpha(w)$ maps the unit circle $\partial\mathbb{D}$ on to the real line $\mathbb{R}$. In fact, for any $w = \mathrm{e}^{i\theta} \in \partial\mathbb{D}$, we have

$$\alpha(\mathrm{e}^{i\theta}) = i\frac{1-\mathrm{e}^{i\theta}}{1+\mathrm{e}^{i\theta}} = \frac{\sin\theta}{1+\cos\theta} \in \mathbb{R}.$$

Therefore, if we denote

$$\alpha(\mathrm{e}^{i\theta}) = i\frac{1+\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}} = x,$$

then we have

$$\mathrm{e}^{i\theta} = \frac{x-i}{x+i}$$

and obtain the following relation between $\mathrm{d}\theta$ and $\mathrm{d}x$ like that

$$\mathrm{d}\theta = \frac{2}{1+x^2}\,\mathrm{d}x \quad \text{or} \quad \mathrm{d}x = \frac{1}{1+\cos\theta}\,\mathrm{d}\theta. \tag{2}$$

Let Φ be the class of all analytic self-mappings of $\mathbb{C}^+$. Each self-mapping φ of $\mathbb{C}^+$ (i.e. $\varphi \in \Phi$) induces a composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for functions f analytic on $\mathbb{C}^+$. It is clear that C_φ takes the spaces of analytic functions on $\mathbb{C}^+$ into itself.

Moreover, for each self-mapping φ of upper half-plane, we obtain a relative self-mapping ϕ of unit disc $\mathbb{D}$ as follows:

$$\phi = \alpha^{-1} \circ \varphi \circ \alpha.$$

For self-mapping $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$, if $\Omega = \varphi(\mathbb{C}^+)$, that is, $\Omega \subset \mathbb{C}^+$ and $\varphi : \mathbb{C}^+ \to \Omega$, we define the norm of the Dirichlet spaces $\mathfrak{D}(\Omega)$ on Ω by

$$\|f\|^2_{\mathfrak{D}(\Omega)} = |f(\varphi(i))|^2 + \int_\Omega |f'(z)|^2 d_{A(z)}.$$

It is easy to see that $\mathfrak{D}(\Omega)$ is also a Hilbert space with inner product:

$$\langle f, g\rangle = f(\varphi(i))\overline{g(\varphi(i))} + \int_\Omega f'(z)\overline{g'(z)} d_{A(z)}.$$

It is not hard to prove that the mapping $Uf = f \circ \varphi$ from Dirichlet spaces $\mathfrak{D}(\Omega)$ to $\mathfrak{D}(\mathbb{C}^+)$ is a unitary operator.

In order to consider the relationship between the Dirichlet spaces and Hardy spaces, it is necessary to introduce the basic definitions of Hardy spaces on the upper half plane.

For $0 < p < \infty$, let $H^p = H^p(\mathbb{C}^+)$ denote the Hardy space of analytic functions f on the upper half plane $\mathbb{C}^+$ satisfying that the quantity

$$\|f\|_{H^p} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

is finite.

Moreover, we still need some knowledge about the subharmonic function and harmonic majorant. A subharmonic function on an open set O in the plane is a function $v : O \to [-\infty, \infty]$ such that

(1) f is upper semicontinuous: $f(z_0) \geq \lim_{z \to z_0} f(z)$ for $z_0 \in O$;
(2) For each $z_0 \in O$ there is $f(z_0) > 0$ such that the disc

$$O(z_0, r(z_0)) = \{z : |z - z_0| < r(z_0)\}$$

is contained in O and for every $r < r(z_0)$, there is

$$f(z_0) \leq \frac{1}{\pi r^2} \int \int_{|z-z_0|<r(z_0)} f(z) \, \mathrm{d}x \, \mathrm{d}y.$$

It is well known that every harmonic function is subharmonic, and for each analytic function $f(z)$ on set O, the function $\log |f(z)|$ is a subharmonic function, moreover, the function $|f(z)|^p = \mathrm{e}^{p \log |f(z)|}$ is also a subharmonic function on O.

The subharmonic function $f(z)$ on O has a harmonic majorant if there is a harmonic function $F(z)$ such that $f(z) \leq F(z)$ throughout set O.

3. Density of rational functions in Dirichlet spaces

As we all know, the polynomials play an indispensable role in Dirichlet spaces $\mathfrak{D}(\mathbb{D})$, because the density of polynomials in $\mathfrak{D}(\mathbb{D})$ is very important. Therefore, finding dense sets in Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ over upper half-plane is often very useful. Especially, considering the density of rational functions in $\mathfrak{D}(\mathbb{C}^+)$ is necessary. However, unlike the classical Dirichlet spaces $\mathfrak{D}(\mathbb{D})$ in which the polynomials are always dense, the dense sets in $\mathfrak{D}(\mathbb{C}^+)$ are not so easy to found. In this section, our main work is to find and construct some dense sets in $\mathfrak{D}(\mathbb{C}^+)$.

In order to obtain the dense sets of $\mathfrak{D}(\mathbb{C}^+)$, and since the conformal mapping α defined in (1) connects the unite disc and upper half-plane, we first have the relationship between $\mathfrak{D}(\mathbb{C}^+)$ and $\mathfrak{D}(\mathbb{D})$ as the following Lemma 3.1. We have to note that the relationship between Dirichlet spaces of unite disc and upper half-plane is different from any other spaces of analytic functions, such as classical Hardy spaces and Bergman spaces, more details about those difference will be discussed in the Section 5.

Lemma 3.1: *An analytic function f on $\mathbb{C}^+$ belongs to $\mathfrak{D}(\mathbb{C}^+)$ if and only if $f \circ \alpha \in \mathfrak{D}(\mathbb{D})$, where α is defined as* (1).

Proof: 'Only if' For any $w \in \mathbb{D}$, there exists a $z \in \mathbb{C}^+$ such that $z = \alpha(w)$. We have

$$\begin{aligned}\int_{\mathbb{D}} |[f \circ \alpha(w)]'|^2 d_{A(w)} &= \int_{\mathbb{D}} |f'(\alpha(w))|^2 |\alpha'(w)|^2 d_{A(w)} \\ &= \int_{\mathbb{C}^+} |f'(z)|^2 d_{A(z)}.\end{aligned}$$

The condition $f \in \mathfrak{D}(\mathbb{C}^+)$ implies the above formula is finite.

Thus, $f \circ \alpha(w) \in \mathfrak{D}(\mathbb{D})$ holds.

Similarly, it is not hard to prove the 'If part' as follows.

'If part:' For any $z \in \mathbb{C}^+$, there exists a $w \in \mathbb{D}$ such that $z = \alpha(w)$. So, we have

$$\begin{aligned}\int_{\mathbb{C}^+} |f'(z)|^2 d_{A(z)} &= \int_{\mathbb{D}} |f'(\alpha(w))|^2 |\alpha'(w)|^2 d_{A(w)} \\ &= \int_{\mathbb{D}} |[f \circ \alpha(w)]'|^2 d_{A(w)}.\end{aligned}$$

Therefore, together with the condition that $f \circ \alpha \in \mathfrak{D}(\mathbb{D})$, we obtain that $f \in \mathfrak{D}(\mathbb{C}^+)$. The proof of Lemma 3.1 is complete. ■

Based on Lemma 3.1, if let the set $\mathbf{R}$ denote all the rational functions in $\mathfrak{D}(\mathbb{C}^+)$, then we can find that the rational functions set $\mathbf{R}$ is dense in $\mathfrak{D}(\mathbb{C}^+)$ and plays the same role as the polynomials in $\mathfrak{D}(\mathbb{D})$, that is the following theorem. Notice that it is not sure all the rational functions in upper half-plane $\mathbb{C}^+$ is dense in $\mathfrak{D}(\mathbb{C}^+)$, but only for the rational functions which belong to $\mathfrak{D}(\mathbb{C}^+)$. For example, $z \bar{\in} \mathfrak{D}(\mathbb{C}^+)$ or $\frac{z^2}{z+1} \bar{\in} \mathfrak{D}(\mathbb{C}^+)$.

Theorem 3.1: *The set* **R** *which contains all the rational functions in Dirichlet spaces* $\mathfrak{D}(\mathbb{C}^+)$ *is dense in* $\mathfrak{D}(\mathbb{C}^+)$.

Proof: Suppose the set **R** is not dense in $\mathfrak{D}(\mathbb{C}^+)$. Then there exists a function $f_0 \in \mathfrak{D}(\mathbb{C}^+)$, for any rational functions sequence $\{r_k\}$ in set **R**, such that

$$\|r_k - f_0\|_{\mathfrak{D}(\mathbb{C}^+)} \nrightarrow 0, \quad (k \to \infty). \tag{3}$$

By $f_0 \in \mathfrak{D}(\mathbb{C}^+)$, and Lemma 3.1, we have $f_0 \circ \alpha \in \mathfrak{D}(\mathbb{D})$.

Since the polynomials are dense in $\mathfrak{D}(\mathbb{D})$, for function $f_0 \circ \alpha(w) \in \mathfrak{D}(\mathbb{D})$, there exists a sequence $\{p_k\}$ of polynomials in $\mathbb{D}$, such that

$$\|p_k - f_o \circ \alpha\|_{\mathfrak{D}(\mathbb{D})} \to 0, \quad (k \to \infty). \tag{4}$$

Moreover, for any $w \in \mathbb{D}$, there exists a $z \in \mathbb{C}^+$ such that $z = \alpha(w)$ and $w = \alpha^{-1}(z)$. From (4), the following holds

$$\|p_k \circ \alpha^{-1} - f_0\|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (k \to \infty). \tag{5}$$

Note that the sequence $\{p_k \circ \alpha^{-1}(z)\}$ is also contained in **R**.

It is clearly, (5) is a contradiction with (3). Therefore, the set **R** is dense in $\mathfrak{D}(\mathbb{C}^+)$. ■

Remark 3.1: Although Theorem 3.1 shows that the rational function set **R** is dense in $\mathfrak{D}(\mathbb{C}^+)$ and also implies that the set **R** will play the same role in $\mathfrak{D}(\mathbb{C}^+)$ as the polynomials play in $\mathfrak{D}(\mathbb{D})$, there is still some difference between $\mathfrak{D}(\mathbb{C}^+)$ and $\mathfrak{D}(\mathbb{D})$. Because the fact that the polynomials is dense in $\mathfrak{D}(\mathbb{D})$ means that all the polynomials are belong to $\mathfrak{D}(\mathbb{D})$ and dense in it, however, not all the rational functions are the functions in $\mathfrak{D}(\mathbb{C}^+)$! So what kind of rational functions belong to $\mathfrak{D}(\mathbb{C}^+)$?

This question will be interest. That is, it is necessary to consider such an important question: What is the special form of the rational functions in the set **R**? We find that more than one set satisfies this conditions. Firstly, from the density of the polynomials in $\mathfrak{D}(\mathbb{D})$ and the relationship between $\mathfrak{D}(\mathbb{C}^+)$ and $\mathfrak{D}(\mathbb{D})$ Lemma 3.1, as well as using Theorem 3.1, we can obtain one dense set of $\mathfrak{D}(\mathbb{C}^+)$ as follows.

Corollary 3.1: *Suppose* α *is the mapping defined in* (1) *from* $\mathbb{D}$ *to* $\mathbb{C}^+$, *if* $\mathbf{R}_\alpha(C^+)$ *is a rational functions set defined as*

$$\mathbf{R}_\alpha(C^+) = \{P \circ \alpha^{-1}(z) : z \in \mathbb{C}^+, P \quad \textit{is any polynomial in the unite disc } \mathbb{D}\}$$

then $\mathbf{R}_\alpha(C^+)$ *is a dense subset of* $\mathfrak{D}(\mathbb{C}^+)$.

Proof: For any rational function $r(z) = p(\alpha^{-1}(z)) \in \mathbf{R}_\alpha(C^+)$, together the relationship of the mapping $\alpha(w)$ and its reverse $\alpha^{-1}(z)$, where $w \in \mathbb{D}$ and $z \in \mathbb{C}^+$, we have $p(w) = r \circ \alpha(w)$ is a polynomial in he unite disc $\mathbb{D}$. By the fact that any polynomial in $\mathbb{D}$ is a function of $\mathfrak{D}(\mathbb{D})$, together with Lemma 3.1, we obtain that the rational function $r(z) = p \circ \alpha^{-1}(z)$ is a function of $\mathfrak{D}(\mathbb{C}^+)$, and then, the set $\mathbf{R}_\alpha(C^+)$ is a subset of $\mathfrak{D}(\mathbb{C}^+)$. Therefore, Theorem 3.1 shows that $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\mathbb{C}^+)$. Thus, the proof of Corollary 3.1 is complete. ■

From Corollary 3.1, we obtain that some rational functions on the upper half-plane such as $P(\alpha^{-1}(z))$ must be the functions of Dirichlet spaces over the upper half-plane. For more general case, what kind of the rational function belongs to the Dirichlet space $\mathfrak{D}(\mathbb{C}^+)$? The second author and her co-workers [12] have considered this type of question for Hardy spaces over tube domains, and obtained that the analytic rational functions with L^p non-tangent boundary limit functions belong to relative Hardy spaces. How about the Dirichlet spaces? Similarly, it will be found that the analytic rational functions with their derivative are L^2 non-tangent boundary limit functions are functions of Dirichlet spaces of upper half-plane.

4. Characterization of composition operators with dense range

After obtaining some dense subset of Dirichlet spaces on upper half-plane in the last section, in what follows, we will solve the main Problem 1.1 for the case of the upper half-plane as Theorem 4.1, which is one characterization of composition operators of $\mathfrak{D}(\mathbb{C}^+)$ with a dense range, and also a generalization of Theorem A which is about the classical case of unit disc.

Theorem 4.1: *Let φ be an analytic self-mapping of $\mathbb{C}^+$ and $\Omega = \varphi(\mathbb{C}^+)$. Then, C_φ reduced by φ is a bounded composition operator on the Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ with dense range if and only if φ is univalent and the rational function set $\mathbf{R}_\alpha(C^+)$ is dense in the Dirichlet spaces $\mathfrak{D}(\Omega)$, where $\mathbf{R}_\alpha(C^+)$ is defined as in Corollary* 3.1.

Proof: 'Only if' First, if C_φ is a bounded composition operator on $\mathfrak{D}(\mathbb{C}^+)$ with dense range, then it is easy to show that φ is univalent. In fact, if φ is not univalent, that is, there are two different points $z_1, z_2 \in \mathbb{C}^+$ such that $\varphi(z_1) = \varphi(z_2)$. Moreover, for $z_1, z_2 \in \mathbb{C}^+$, there exist two different points $w_1 \in \mathbb{D}$ and $w_2 \in \mathbb{D}$ such that $w_1 = \alpha^{-1}(z_1)$ and $w_2 = \alpha^{-1}(z_2)$. For any function $f(z) \in \mathfrak{D}(\mathbb{C}^+)$, we have $C_\varphi f(z_1) = f(\varphi(z_1))$ and $C_\varphi f(z_2) = f(\varphi(z_2))$. So that the following formula holds:

$$C_\varphi f(z_1) = C_\varphi f(z_2). \tag{6}$$

By $f(z) \in \mathfrak{D}(\mathbb{C}^+)$ and Lemma 3.1, we have $f \circ \alpha(w) \in \mathfrak{D}(\mathbb{D})$. Together with formula (6), we have

$$C_\varphi(f \circ \alpha)(w_1) = C_\varphi(f \circ \alpha)(w_2). \tag{7}$$

However, since C_φ has a dense range in $\mathfrak{D}(\mathbb{C}^+)$, for any function $f(z) \in \mathfrak{D}(\mathbb{C}^+)$, there exist a sequence $\{f_n(z)\} \subset \mathfrak{D}(\mathbb{C}^+)$ such that

$$\|C_\varphi f_n - f\|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (n \to \infty).$$

By Lemma 3.1, there are $\{f_n \circ \alpha(w)\} \subset \mathfrak{D}(\mathbb{D})$ and $f \circ \alpha(w) \in \mathfrak{D}(\mathbb{D})$. Thus, there is

$$\|C_\varphi(f_n \circ \alpha) - f \circ \alpha\|_{\mathfrak{D}(\mathbb{D})} \to 0, \quad (n \to \infty).$$

If we take $f \circ \alpha(w) = w$, then $w_1 = w_2$, moreover, there hold

$$z_1 = \alpha(w_1) = \alpha(w_2) = z_2,$$

which is a contradiction with the condition $z_1 \neq z_2$. So that the mapping φ is univalent.

Next we are to prove the set $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\Omega)$. If not so, then for any sequence $\{r_k\}$ of rational functions in $\mathbf{R}_\alpha(C^+)$, there exists a function $g_0 \in \mathfrak{D}(\Omega)$ such that,

$$\|r_k - g_0\|_{\mathfrak{D}(\Omega)} \nrightarrow 0, \quad (k \to \infty). \tag{8}$$

Thus, there is

$$\|C_\varphi r_k - C_\varphi g_0\|_{\mathfrak{D}(\mathbb{C}^+)} \nrightarrow 0, \quad (k \to \infty). \tag{9}$$

If let $f_0 = C_\varphi g_0$, since C_φ has a dense range in $\mathfrak{D}(\mathbb{C}^+)$, then there is a sequence $\{f_k(z)\}$ of functions in $\mathfrak{D}(\mathbb{C}^+)$ such that

$$\|C_\varphi f_k - f_0\|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (k \to \infty), \tag{10}$$

where $f_k(z) = C_\varphi r_k = r_k \circ \varphi(z)$.

By Theorem 3.1, the set $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\mathbb{C}^+)$, so that, for any function $f \in \mathfrak{D}(\mathbb{C}^+)$, there exists a sequence in $\mathbf{R}_\alpha(C^+)$ that converges to f in $\mathfrak{D}(\mathbb{C}^+)$ norm. Therefore, by $f_0 \in \mathfrak{D}(\mathbb{C}^+)$, there is a sequence $\{r_k\}$ in $\mathbf{R}_\alpha(C^+)$ such that

$$\|C_\varphi r_k - f_0\|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (k \to \infty).$$

However, the set $\mathbf{R}_\alpha(C^+)$ is contained in Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$, and then the rational function sequence $\{r_k\}$ is also in $\mathfrak{D}(\mathbb{C}^+)$. Obviously, this is a contradiction with (9) and (10). Therefore, the rational function set $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\Omega)$.

'If part' If φ is univalent and $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\Omega)$, we are to prove that the range, $R(C_\varphi)$, of composition operator C_φ, is dense in $\mathfrak{D}(\mathbb{C}^+)$.

First the condition that φ is univalent shows that C_φ is a univalent operator from $\mathfrak{D}(\Omega)$ to $\mathfrak{D}(\mathbb{C}^+)$ with the inverse C_φ^{-1} from $\mathfrak{D}(\mathbb{C}^+)$ to $\mathfrak{D}(\Omega)$. That is, for any function $g \in \mathfrak{D}(\mathbb{C}^+)$, there exists a function $f \in \mathfrak{D}(\Omega)$ such that $C_\varphi f = g$.

Since $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\Omega)$, there exists a rational function sequence $\{r_k\}$ contained in $\mathbf{R}_\alpha(C^+)$ such that

$$\|r_k - f\|_{\mathfrak{D}(\Omega)} \to 0, \quad (k \to \infty).$$

And then

$$\|C_\varphi r_k - C_\varphi f\|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (k \to \infty).$$

Together with the relationship $C_\varphi f = g$, we have

$$\|C_\varphi r_k - g|_{\mathfrak{D}(\mathbb{C}^+)} \to 0, \quad (k \to \infty).$$

Moreover, note that $\mathbf{R}_\alpha(C^+)$ is also contained in $\mathfrak{D}(\mathbb{C}^+)$, thus $\{r_k\}$ is also contained in $\mathfrak{D}(\mathbb{C}^+)$, hence, $R(C_\varphi)$ is dense in $\mathfrak{D}(\mathbb{C}^+)$. Therefore, the proof of Theorem 4.1 is complete. ■

It should be pointed out that there is some differences between Theorem A and Theorem 4.1, separately about cases of the unit disc and the upper half-plane. For example, Theorem A can associate with Dirichlet spaces $\mathfrak{D}(\mathbb{D})$ and Bergman spaces $A^2(\Omega)$, but

Theorem 4.1 cannot achieve that. This shows that the properties of composition operators on $\mathfrak{D}(\mathbb{D})$ are richer than $\mathfrak{D}(\mathbb{C}^+)$.

It is not hard to explain this difference. In fact, for the case of unit disc, as we all know, the polynomials play a very important role and they both belong to the spaces $\mathfrak{D}(\mathbb{D})$ and $A^2(\mathbb{D})$. Moreover, since the derivative of any polynomial is still a polynomial, the polynomials are dense in $A^2(\Omega)$ if and only if they are dense in $\mathfrak{D}(\Omega)$, where $\Omega \subseteq \mathbb{D}$. Therefore, Theorem A characterizes the composition operators of $\mathfrak{D}(\mathbb{D})$ by using the relationship between $\mathfrak{D}(\mathbb{D})$ and $A^2(\Omega)$. However, the case of the half-plane could not. Because for any rational functions $r(z)$ of $\mathfrak{D}(\mathbb{C}^+)$ on the upper half-plane, we are not sure that $r(z)$ is also a function of Bergman spaces $A^2(\mathbb{C}^+)$. Therefore, under the same conditions of Theorem 4.1, the density of the set $\mathbf{R}_\alpha(C^+)$ in Bergman spaces $A^2(\Omega)$ could not hold.

Next, we continue to consider the boundedness of composition operators on $\mathfrak{D}(\mathbb{C}^+)$, and obtain the following result. Let us first recall a simple but important property of composition operators on Dirichlet spaces as the following Lemma 4.1.

Lemma 4.1 ([13]): *Let D_1 and D_2 be domains in complex plane. If $\varphi : D_1 \to D_2$ is univalent and $f : D_2 \to \mathbb{C}$ is a holomorphic function, then*

$$\int_{D_1} |(f \circ \varphi)'(z)|^2 d_{A(z)} = \int_{D_2} |f'(w)|^2 d_{A(w)}.$$

In particular, taking $D_1 = \mathbb{C}^+$ and $D_2 = \varphi(\mathbb{C}^+) = \Omega$, we can obtain the other form of the norm of $\mathfrak{D}(\Omega)$ as

$$\|C_\varphi f\|^2_{\mathfrak{D}(\Omega)} = |f(\varphi(i))|^2 + \int_\Omega |f'(w)|^2 d_{A(w)} = |f(\varphi(i))|^2 + \int_{\mathbb{C}^+} |(f \circ \varphi)'(z)|^2 d_{A(z)}.$$

Based on the last lemma, we obtain the following result.

Theorem 4.2: *If the self-mapping $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ is univalent, then the composition operator C_φ is self-mapping from $\mathfrak{D}(\mathbb{C}^+)$ to $\mathfrak{D}(\mathbb{C}^+)$ and bounded.*

***Proof*:** Let $\Omega = \varphi(\mathbb{C}^+) \subseteq \mathbb{C}^+$, for any function f of $\mathfrak{D}(\mathbb{C}^+)$, and by Lemma 4.1, then we have

$$\begin{aligned}
\|C_\varphi f\|^2_{\mathfrak{D}(\mathbb{C}^+)} &= |f(\varphi(i))|^2 + \int_{\mathbb{C}^+} |(f \circ \varphi)'(z)|^2 d_{A(z)} \\
&= |f(\varphi(i))|^2 + \int_\Omega |f'(w)|^2 d_{A(w)} \\
&\le |f(\varphi(i))|^2 + \int_{\mathbb{C}^+} |f'(w)|^2 d_{A(w)}.
\end{aligned}$$

Since $f \in \mathfrak{D}(\mathbb{C}^+)$, there is

$$\int_{\mathbb{C}^+} |f'(w)|^2 d_{A(w)} < \infty.$$

Therefore,

$$\|C_\varphi f\|^2_{\mathfrak{D}(\mathbb{C}^+)} < \infty,$$

that is, $C_\varphi f$ maps $\mathfrak{D}(\mathbb{C}^+)$ back into itself.

Moreover, by the closed graph theorem, we obtain that $C_\varphi f$ is bounded on $\mathfrak{D}(\mathbb{C}^+)$. Thus, the proof of Theorem 4.2 is complete. ■

In what following, we obtain some other properties of composition operators on $\mathfrak{D}(\mathbb{C}^+)$.

Theorem 4.3: *Let $\Omega \subseteq \mathbb{C}^+$ be a simply connect domain and the mapping $\varphi : \mathbb{C}^+ \to \Omega$ be univalent. If the Lebesgue measure of the complement of Ω in $\mathbb{C}^+$ is zero ($m(\mathbb{C}^+ \setminus \Omega) = 0$), then the composition operator $C_\varphi : \mathfrak{D}(\mathbb{C}^+) \to \mathfrak{D}(\mathbb{C}^+)$ is bounded and has a close range.*

Proof: Since the mapping φ is univalent, Theorem 4.2 shows that the composition operator $C_\varphi : \mathfrak{D}(\mathbb{C}^+) \to \mathfrak{D}(\mathbb{C}^+)$ is bounded.

Consider the norms of $\mathfrak{D}(\mathbb{C}^+)$ and $\mathfrak{D}(\Omega)$ respectively as follows:

$$\|f\|^2_{\mathfrak{D}(\mathbb{C}^+)} = |f(i)|^2 + \int_{\mathbb{C}^+} |f'(z)|^2 d_{A(z)}$$

and

$$\|f\|^2_{\mathfrak{D}(\Omega)} = |f(\varphi(i))|^2 + \int_{\Omega} |f'(z)|^2 d_{A(z)}.$$

By the condition $m(\mathbb{C}^+ \setminus \Omega) = 0$, we obtain the following formula for any function f of $\mathfrak{D}(\mathbb{C}^+)$:

$$\int_{\mathfrak{D}(\mathbb{C}^+)} |f'(z)|^2 d_{A(z)} = \int_{\Omega} |f'(z)|^2 d_{A(z)} + \int_{\mathbb{C}^+\setminus\Omega} |f'(z)|^2 d_{A(z)} = \int_{\Omega} |f'(z)|^2 d_{A(z)}.$$

Therefore, it is easy to prove that the two norms $\|f\|^2_{\mathfrak{D}(\mathbb{C}^+)}$ and $\|f\|^2_{\mathfrak{D}(\Omega)}$ are equivalent on $\mathfrak{D}(\mathbb{C}^+)$. Since the rational function set $\mathbf{R}_\alpha(C^+)$ defined in Corollary 3.1 is contained in $\mathfrak{D}(\mathbb{C}^+)$, and $\mathfrak{D}(\mathbb{C}^+)$ is a subspace of $\mathfrak{D}(\Omega)$, we know that $\mathbf{R}_\alpha(C^+)$ is also contained in $\mathfrak{D}(\Omega)$.

Moreover, since $\mathbf{R}_\alpha(C^+)$ is dense in $\mathfrak{D}(\mathbb{C}^+)$, the closure of $\mathbf{R}_\alpha(C^+)$ is just $\mathfrak{D}(\mathbb{C}^+)$, thus $\mathfrak{D}(\mathbb{C}^+)$ is a closed subspace of $\mathfrak{D}(\Omega)$. We note that the mapping $C_\varphi f = f \circ \varphi : \mathfrak{D}(\Omega) \to \mathfrak{D}(\mathbb{C}^+)$ is unitary operator. Therefore, the range of C_φ is closed. ■

5. The relationship between Dirichlet spaces and Hardy spaces on half-plane

From Lemma 3.1 of Section 3, we obtain the relationship between the Dirichlet spaces $\mathfrak{D}(\mathbb{D})$ and $\mathfrak{D}(\mathbb{C}^+)$ by making essential use of the linear fractional map $\alpha(w) = i\frac{1-w}{1+w}$ which transforms unit disc onto the upper-half plane.

There are some remarks we might make. The 'If and only if' relationship between the Dirichlet spaces for the disc and half-plane in Lemma 3.1 only holds for Dirichlet spaces but not for any other analytic function spaces, such as Hardy spaces and Bergman spaces. Of course, all the differences are due to their different definitions.

For example, especially for classical Hardy spaces H^p, $0 < p < \infty$, the relationship between Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{C}^+)$ is not so simple and beautiful as Dirichlet spaces in Lemma 3.1. There are many discussions about the relation between the Hardy spaces

for the disc and half-plane, such as Cima [14], Garnett [15] and Hoffman [16]. In [14], it was shown that any analytic function f of upper half-plane $\mathbb{C}^+$ belongs to Hardy spaces $H^p(\mathbb{C}^+)$ if and only if $(f\circ\alpha)(\alpha')^{1/p}\in H^p(\mathbb{D})$, where α is defined as (1). In [15], only the 'only if part' as in Lemma 3.1 holds for Hardy spaces, that is, if $f(z)\in H^p(\mathbb{C}^+)$ then $f\circ\alpha(w)\in H^p(\mathbb{D})$, the converse is not true. We still take some information from [15,16] which imply that $f(w)\in H^p(\mathbb{D}$ if and only if $\frac{\pi^{-1/p}}{(z+i)^{2/p}}f(\alpha^{-1}(z))\in H^p(\mathbb{C}^+)$.

In this section, we mainly consider the relationship between Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ and classical Hardy spaces $H^2(\mathbb{C}^+)$ of half-plane. It is well known, for the case of unit disc, the following relationship between $\mathfrak{D}(\mathbb{D})$ and $H^2(\mathbb{D})$ holds.

Lemma 5.1: *The Dirichlet spaces $\mathfrak{D}(\mathbb{D})$ is contained in Hardy spaces $H^2(\mathbb{D})$.*

It is natural to ask how about the relationship between Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ and Hardy spaces $H^2(\mathbb{C}^+)$ on the upper half-plane, is it the same as the case of unit disc? We find that the answer is 'no' and obtain a similar result such as Theorem 5.1, which is simple but very useful for the theory of Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$. Notice that the proof of Theorem 5.1 is different from the proof of Lemma 5.1, because the unboundedness of upper half-plane restricts that we could not define the Dirichlet spaces $\mathfrak{D}(\mathbb{C}^+)$ by the Taylor coefficients like $\mathfrak{D}(\mathbb{D})$. Therefore, we have to find some other methods to prove it, although it is not difficult. Moreover, in order to prove Theorem 5.1, we will make use of the previous stated relationship between $H^2(\mathbb{D})$ and $H^2(\mathbb{C}^+)$, which is the bridge between the case of unit disc and upper half-plane.

More precisely, we need the following lemmas.

Lemma 5.2 ([15]): *If $f(z)$ is an analytic function in upper half-plane such that the subharmonic function $|f(z)|^p$, $0<p<\infty$ has a harmonic majorant, then*

$$F(z)=\frac{\pi^{-1/p}}{(z+i)^{2/p}}f(z)$$

is in Hardy spaces $H^p(\mathbb{C}^+)$.

Lemma 5.3 ([15]): *Let $g(w)$ be a subharmonic function in the unit disc. Then $g(w)$ has a harmonic majorant if and only if*

$$\sup_{0\le r\le 1}\int g(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta<\infty.$$

Clearly, Lemma 5.3 implies that for function $g(w)$ of Hardy spaces $H^p(\mathbb{D})$, $|g(w)|^p$ has a harmonic majorant.

Based on the above lemma results, we are now to obtain the following relationship between the Hardy spaces $H^2(\mathbb{C}^+)$ and Dirichlet space $\mathfrak{D}(\mathbb{C}^+)$.

Theorem 5.1: *If the analytic function $f(z)\in\mathfrak{D}(\mathbb{C}^+)$, then $f(z)[(\alpha^{-1}(z))']^{1/2}\in H^2(\mathbb{C}^+)$, that is, $\frac{\sqrt{-2i}}{z+i}f(z)\in H^2(\mathbb{C}^+)$, where α is defined as* (1).

Proof: On one hand, for any function $f(z)\in\mathfrak{D}(\mathbb{C}^+)$, it is easy to obtain that the functions $f(z)[(\alpha^{-1}(z))']^{1/2}$ and $\frac{\sqrt{-2i}}{z+i}f(\alpha^{-1}(z))$ are also analytic on upper half-plane $\mathbb{C}^+$.

On the other hand, by $f(z) \in \mathfrak{D}(\mathbb{C}^+)$ and Lemma 3.1, we have $f \circ \alpha(w) \in \mathfrak{D}(\mathbb{D})$. Consequently, Lemma 5.1 shows that $f \circ \alpha(w) \in H^2(\mathbb{D})$.

Moreover, Lemma 5.3 induces that $|f \circ \alpha(w)|^p$ has a harmonic majorant $u(w)$. Hence, $|f \circ \alpha(\alpha^{-1}(z))|^p = |f(z)|^p$ has a harmonic majorant $u(\alpha(z))$.

Therefore, by Lemma 5.2, we obtain that $\frac{\pi^{-1/2}}{z+i} f(z) \in H^2(\mathbb{C}^+)$. So that, there also holds $\frac{\sqrt{-2i}}{z+i} f(z) \in H^2(\mathbb{C}^+)$.

At last, we note that $(\alpha^{-1})'(z) = \frac{-2i}{(z+i)^2}$, then the relationship

$$f(z)[(\alpha^{-1}(z))']^{1/2} \in H^2(\mathbb{C}^+)$$

is also established. ■

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

Cao is supported by NSF of China [Grant No. 12071155]; Li is supported by NSF of China [Grant No. 11901205] and NSF of China [Grant No.11901205].

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Research Article

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Toeplitz operators on weighted Bergman spaces over tubular domains

https://doi.org/10.1515/gmj-2025-2001
Received April 27, 2024; revised October 1, 2024; accepted November 6, 2024

Abstract: In this paper, we mainly study the necessary and sufficient conditions for the boundedness and compactness of Toeplitz operators on weighted Bergman spaces over a tubular domains by using the Carlson measures on tubular domains. We also give some related results about Carlson measures.

Keywords: Weighted Bergman spaces, Toeplitz operator, tubular domain, Carleson measure

MSC 2020: 47A05, 46B99, 32A36, 30H20, 32A07

1 Introduction

Bergman's book [5] provides the first systematic discussion of the Hilbert space of square-integrable analytic functions on the domain, now known as the Bergman space defined on a domain. The Bergman space is a closed subspace of the familiar L^p space. When $p = 2$, the Bergman space is a Hilbert space. A useful tool for studying the Bergman space is the reproducing kernel, which plays a very important role. For related theories on the reproducing kernel, please refer to relevant literature [4, 5, 10, 17].

Another important tool in the study of operators on function spaces is the Carleson measure, which was initially introduced by Carleson [6] to address the Corona problem. Nowadays, Carleson measures play a crucial role in studying the boundedness and compactness of operators, especially for Toeplitz operators, see [19, 20]. Regarding the further applications of Carleson measures in operator theory on function spaces, refer to [3, 7, 11, 16]. The theory of Toeplitz operators on the unit disk and the unit ball in the Bergman spaces has been extensively studied by many authors, such as (see, e.g., [19, 20]. Subsequently, many authors have also extended the domains to bounded symmetric domains [21], strongly pseudoconvex domains [1, 2, 12], pseudoconvex domains [13], and so on.

However, research on the theory of the Bergman spaces on unbounded domains is scarce. In particular, when $n = 1$, the Bergman spaces on the upper half-plane lack many good properties of the Bergman spaces over the unit disk, e.g. it is the well-known fact that constant functions and monomial functions do not belong to Bergman spaces over the upper half-plane.

In the present paper, we are interested in the case of the higher-dimensional unbounded domains, such as the tubular domains. More specifically, this paper will mainly study the boundedness and compactness of Toeplitz operators on a certain class of tubular domains in $\mathbb{C}^n$ and their relationship with Carleson measures. These tubular domains may share some similarities with the well-known second kind of Siegel upper half-space, but the second kind of Siegel upper half-spaces are not tubes in $\mathbb{C}^n$. Instead, the tubes are larger than the second kind of Siegel upper half-space, hence they have corresponding research value.

Deng, Huang and Qianet [8] computed the reproducing kernel of the Bergman spaces on such tubes using Laplace transform methods, laying the groundwork for subsequent theoretical research. Liu, Deng and Yin [15]

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provided some basic properties of the Bergman spaces on these tubes. Si, Liu and Si [14] studied the boundedness and compactness of Toeplitz operators on the Bergman space of the second kind of Siegel upper half-space and their relationship with Carleson measures. This paper will mainly study the theory of Toeplitz operators on the Bergman spaces over tubes with the help of Carleson measure, which is both a powerful tool and an interesting object to study.

The structure of the paper is as follows. Section 2 provides an overview of fundamental terminology. Section 3 presents essential lemmas and their proofs. In Section 4, we obtain the characterization of Carleson measures on the tubular domains. Moving forward, in Section 5, we can find a dense subspace of Bergman spaces over tubular domains, which is crucial for establishing the boundedness of Toeplitz operators. Finally, Section 6 comprehensively explores the characterizations of boundedness and compactness of Toeplitz operators on Bergman spaces over tubular domains, with detailed discussions on Theorem 6.1 and Theorem 6.2.

2 Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For any two points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we write

$$z \cdot \bar{w} := z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$
$$z^2 = z \cdot z := z_1^2 + z_2^2 + \cdots + z_n^2,$$
$$|z| := \sqrt{z \cdot \bar{z}} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The set $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ will be called the unit ball of $\mathbb{C}^n$.

The tubular domain T_B of $\mathbb{C}^n$ with base B is defined as follows:

$$T_B = \{z = x + iy \in \mathbb{C}^n : x \in \mathbb{R}^n,\ y \in B \subseteq \mathbb{R}^n\},$$

where

$$B = \{(y', y_n) = (y_1, \dots, y_{n-1}, y_n) \in \mathbb{R}^n : y'^2 := y_1^2 + \cdots + y_{n-1}^2 < y_n\}.$$

First, we review the Bergman metric on domains in $\mathbb{C}^n$. In the same way as for the case of a unit ball, let $K(z, w)$ be the Bergman kernel of T_B. We define the complex matrix

$$\mathbf{B}(z) = (b_{ij}(z))_{1\le i,j\le n} = \frac{1}{n+1}\Big(\frac{\partial^2}{\partial \bar{z}_i \partial z_j} \ln K(z, z)\Big)_{1\le i,j\le n}$$

as the Bergman matrix of T_B.

For a smooth curve $\gamma : [0, 1] \to T_B$, we define

$$l(\gamma) = \int_0^1 \langle \mathbf{B}(\gamma(t))\gamma'(t), \gamma'(t)\rangle\, dt.$$

Based on the definition of $l(\gamma)$, we can define the Bergman metric β on T_B as follows:

$$\beta(z, w) = \inf\{l(\gamma) : \gamma(0) = 1,\ \gamma(1) = w\}.$$

Let $D(z, r)$ denote the Bergman metric ball at z with radius r, that is,

$$D(z, r) = \{w \in T_B : \beta(z, w) < r\}.$$

It should be noted that the definitions of the Bergman matrix and the Bergman disk on the unit ball $\mathbb{B}_n$ are derived from the reference [19], and the similar definitions on the tubular domain T_B are derived from the reference [15].

We define the space $L^p_\alpha(T_B)$, which is composed of all Lebesgue measurable functions f on T_B, and its norm

$$\|f\|_{p,\alpha} = \left\{\int_{T_B} |f(z)|^p\, dV_\alpha(z)\right\}^{\frac{1}{p}}$$

is finite, where $dV_\alpha(z) = (y_n - |y'|^2)^\alpha\, dV(z)$, $\alpha > -1$, $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$.

The Bergman space $A^p_\alpha(T_B)$ on tube T_B is a set composed of all holomorphic functions in $L^p_\alpha(T_B)$.

On the Bergman space $A^p_\alpha(T_B)$, every valuation functional is a bounded linear functional. Specifically, for each function $f \in A^p_\alpha(T_B)$, by the subharmonicity of $|f|^p$, we have

$$|f(z)|^p \le \frac{C}{\rho(z)^{n+\alpha+1}} \int_{D(z,r)} |f(w)|^p \, dV_\alpha(w) \le C \frac{\|f\|^p_{p,\alpha}}{\rho(z)^{n+\alpha+1}}, \tag{2.1}$$

where $0 < p < \infty$, $\alpha > -1$, $r > 0$ and C is a positive constant.

So, according to the Weierstrass theorem, it is easy to obtain that the Bergman space $A^p_\alpha(T_B)$ is the closed subspace of $L^p_\alpha(T_B)$. At the same time, we know that when $1 \le p < \infty$, the space $A^p_\alpha(T_B)$ is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. In particular, when $p = 2$, $A^2_\alpha(T_B)$ is a Hilbert space.

According to the classical theory of Hilbert spaces in functional analysis, we know that there is an orthogonal projection P_α from $L^2_\alpha(T_B)$ to $A^2_\alpha(T_B)$. It is the following integral operator:

$$P_\alpha f(z) = \int_{T_B} K_\alpha(z, w) f(w) \, dV_\alpha(w),$$

with the Bergman kernel

$$K_\alpha(z, w) = \frac{2^{n+1+2\alpha}\Gamma(n+1+\alpha)}{\Gamma(\alpha+1)\pi^n}\big((z' - \overline{w'})^2 - 2i(z_n - \overline{w_n})\big)^{-n-\alpha-1}.$$

The explicit expression of the Bergman kernel on the tubular domain T_B is derived from [8].

For convenience, we introduce the following notation:

$$\rho(z, w) = \frac{1}{4}\big((z' - \overline{w'})^2 - 2i(z_n - \overline{w_n})\big),$$

and let $\rho(z) := \rho(z, z) = y_n - y'^2$.

With the above notion of $\rho(z, w)$, the weighted Bergman kernel of T_B becomes

$$K_\alpha(z, w) = \frac{\Gamma(n+\alpha+1)}{2^{n+1}\pi^n\Gamma(\alpha+1)\rho(z, w)^{n+\alpha+1}}.$$

Recall $\rho(z) = y_n - y'^2$ and let $\partial T_B := \{z \in \mathbb{C}^n : \rho(z) = 0\}$ denote the boundary of T_B and $\partial\widehat{T_B} := \partial T_B \cup \{\infty\}$.

We consider $\lim_{z\to\partial\widehat{T_B}} f(z) = 0$ if $f(z) \to 0$ as $\rho(z) \to 0^+$and $f(z) \to 0$ as $|z| \to \infty$. This is equivalent when for all $\varepsilon > 0$, there is a compact set $K \subset T_B$ such that $\sup_{z\in T_B\setminus K} |f(z)| < \varepsilon$.

We denote by $C_0(T_B)$ the space of complex-valued continuous functions f on T_B such that $f(z) \to 0$ as $z \to \partial\widehat{T_B}$.

Next, we define two kinds of important functions, which will be related to our main Theorems 6.1 and 6.2.

For a positive Borel measure μ on T_B, we define a function $\tilde{\mu}$ on T_B by

$$\tilde{\mu}(z) := \int_{T_B} |k_z(w)|^2 d\mu(w), \quad z \in T_B,$$

where, for fixed $z \in T_B$,

$$k_z(w) := K(z, w)/\sqrt{K(z, z)}, \quad w \in T_B.$$

For $z \in T_B$ and $r > 0$, we define the averaging function

$$\hat{\mu}_r(z) := \frac{\mu(D(z, r))}{V_\alpha(D(z, r))}.$$

Finally, we give a biholomorphic mapping Φ between the unit ball $\mathbb{B}_n$ and the tubular domain T_B, which indicates that the two are holomorphically equivalent. This mapping Φ is mentioned in [15], but the author does not give where it comes from. In fact, it comes from [9]. In [9], the author introduces three domains, including the unit ball, the Siegel domain and the tubular domain mentioned in this paper. The author also gives the holomorphic mapping between the Siegel domain and the tubular domain. Therefore, through a simple composite

operation, the expression of Φ is obtained as follows:

$$\Phi(z) = \Big(\frac{\sqrt{2}z'}{1+z_n}, i\frac{1-z_n}{1+z_n} - i\frac{z'\cdot z'}{(1+z_n)^2}\Big), \quad z \in \mathbb{B}_n,$$

and it is not hard to calculate that

$$\Phi^{-1}(w) = \Big(\frac{2iw'}{i+w_n+\frac{i}{2}w'\cdot w'}, \frac{i-w_n-\frac{i}{2}w'\cdot w'}{i+w_n+\frac{i}{2}w'\cdot w'}\Big), \quad w \in T_B.$$

Throughout the paper, we use C to denote a positive constant whose value may change from line to line but does not depend on the functions being considered. The notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$, and the notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

3 Main lemmas

To prove our main results, we need the following key lemmas, where Lemmas 3.1–3.4 are from [15]. They play a crucial role as instrumental lemmas in the present paper.

Lemma 3.1. *There exists a positive integer N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}$ in T_B with the following properties:*
(1) $T_B = \bigcup_{k=1}^{\infty} D(a_k, r)$,
(2) *the sets $D(a_k, \frac{r}{4})$ are mutually disjoint,*
(3) *each point $z \in T_B$ belongs to at most N of the sets $D(a_k, 2r)$.*
The above sequence $\{a_k\}$ is also called the r-lattice.

Lemma 3.2. *For any $r > 0$, the inequalities*

$$|\rho(z,u)| \simeq |\rho(z,v)|$$

hold for all $z, u, v \in T_B$ with $\beta(u,v) < r$.

Lemma 3.3. *For any $z \in T_B$ and $r > 0$, we have*

$$V_\alpha(D(z,r)) \simeq \rho(z)^{n+\alpha+1}.$$

Lemma 3.4. *Let $r, s > 0$, $t > -1$, and $r + s - t > n + 1$. Then*

$$\int_{T_B} \frac{\rho(w)^t}{\rho(z,w)^r \rho(w,u)^s}\, dV(w) = \frac{C_1(n,r,s,t)}{\rho(z,u)^{r+s-t-n-1}}$$

for all $z, u \in T_B$, where

$$C_1(n,r,s,t) = \frac{2^{n+1}\pi^n \Gamma(1+t)\Gamma(r+s-t-n-1)}{\Gamma(r)\Gamma(s)}.$$

In particular, let $s, t \in \mathbb{R}$, if $t > -1$, $s - t > n + 1$, then

$$\int_{T_B} \frac{\rho(w)^t}{|\rho(z,w)|^s}\, dV(w) = \frac{C_1(n,s,t)}{\rho(z)^{s-t-n-1}}.$$

Otherwise, the above equation is infinity.

In order to facilitate the subsequent proof of the theorem, the following lemma gives the real Jacobian determinant of the holomorphic map Φ^{-1} at $z \in T_B$.

Lemma 3.5. *The real Jacobian of Φ^{-1} at $z \in T_B$ is*

$$(J_R\Phi^{-1})(z) = \frac{1}{4|\rho(z,\mathbf{i})|^{2(n+1)}},$$

where $\mathbf{i} := (0', i) \in T_B$.

Proof. Simple calculations show that

$$\rho(z,\mathbf{i}) = \frac{1}{4}(z'^2 - 2iz_n + 2),$$

so

$$(J_C\Phi^{-1})(z) = \begin{vmatrix} \frac{2\rho(w,\mathbf{i})-w_1^2}{2\rho(w,\mathbf{i})^2} & \frac{-w_1w_2}{2\rho(w,\mathbf{i})^2} & \cdots & \frac{-w_1w_{n-1}}{2\rho(w,\mathbf{i})^2} & \frac{iw_1}{2\rho(w,\mathbf{i})^2} \\ \frac{-w_2w_1}{2\rho(w,\mathbf{i})^2} & \frac{2\rho(w,\mathbf{i})-w_2^2}{2\rho(w,\mathbf{i})^2} & \cdots & \frac{-w_2w_{n-1}}{2\rho(w,\mathbf{i})^2} & \frac{iw_2}{2\rho(w,\mathbf{i})^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-w_{n-1}w_1}{2\rho(w,\mathbf{i})^2} & \frac{-w_{n-1}w_2}{2\rho(w,\mathbf{i})^2} & \cdots & \frac{2\rho(w,\mathbf{i})-w_{n-1}^2}{2\rho(w,\mathbf{i})^2} & \frac{iw_{n-1}}{2\rho(w,\mathbf{i})^2} \\ \frac{-w_1}{2\rho(w,\mathbf{i})^2} & \frac{-w_2}{2\rho(w,\mathbf{i})^2} & \cdots & \frac{-w_{n-1}}{2\rho(w,\mathbf{i})^2} & \frac{i}{2\rho(w,\mathbf{i})^2} \end{vmatrix}$$
$$= \frac{i}{2\rho(z,\mathbf{i})^{n+1}}.$$

Therefore, $(J_R\Phi^{-1})(z) = |(J_C\Phi^{-1})(z)|^2 = \frac{1}{4|\rho(z,\mathbf{i})|^{2(n+1)}}$. □

The following lemma plays a key role in estimating the inequality.

Lemma 3.6. *For any $z, w \in T_B$, we have*

$$2|\rho(z,w)| \geq \max\{\rho(z), \rho(w)\}.$$

Proof. Let $\delta_t(w) = (tw', t^2w_n)$ be the nonisotropic dilation for $t > 0$, $w \in T_B$. For any $w \in T_B$ and each fixed $z \in T_B$, consider the holomorphic mapping

$$h_z(w) := \left(w' - z', w_n - \operatorname{Re} z_n - iw'\overline{z'} + \frac{i|z'|^2}{2} + \frac{i\overline{z'}\cdot\overline{z'}}{4} + \frac{iw'\cdot z'}{4}\right).$$

It is evident that $h_z(w)$ is a holomorphic automorphism of T_B. Thus, the mapping $\sigma_z := \delta_{\rho(z)^{-\frac{1}{2}}} \circ h_z$ is a holomorphic automorphism of T_B. Simple calculations reveal that $\sigma_z(z) = \mathbf{i} := (0', i)$ and

$$(J_{\mathbb{C}}\sigma_z)(w) = \rho(z)^{-\frac{n+1}{2}},$$

where $(J_{\mathbb{C}}\sigma_z)(w)$ denotes the complex Jacobian of σ_z at w. Hence, we obtain

$$\begin{aligned} K_\alpha(z,w) &= (J_{\mathbb{C}}\sigma_z)(z)K_\alpha(\sigma_z(z),\sigma_z(w))\overline{(J_{\mathbb{C}}\sigma_z)(w)} \\ &= K_\alpha(\mathbf{i},\sigma_z(w))\ \rho(z)^{-n-1}. \end{aligned}$$

It is worth noting that

$$|K_\alpha(\mathbf{i},w)| \leq \frac{\Gamma(n+\alpha+1)2^\alpha}{\pi^n\Gamma(\alpha+1)}$$

for all $w \in T_B$. By substituting the expression for $K_\alpha(z,w)$ into the above inequality, considering the arbitrariness of α and n, and rearranging the positions of z and w, we arrive at the desired result. □

As is well known, the reproducing kernel is an element of the Bergman space $A^2_\alpha(T_B)$, hence its norm is finite. Lemma 3.7 below clarifies the size of the reproducing kernel.

Lemma 3.7. *For $1 < p < \infty$ and $\alpha > -1$, for each $z \in T_B$, the Bergman kernel function $K_{\alpha,z}(w)$ belongs to $A^p_\alpha(T_B)$, and its norm is $C\rho(z)^{-\frac{1}{p'}(n+\alpha+1)}$, where p' is the conjugate exponent of p and C is a positive constant depending only on n, α and p.*

Proof. The above conclusion can be obtained through straightforward calculation and Lemma 3.4, so we omit the proof. □

The following lemma provides an equivalent condition for the weak convergence of functions in Bergman spaces. Although it is well-known, its proof has not been found, therefore, we provide one below.

Lemma 3.8. *Suppose $\{f_j\}$ is a sequence in $A^p_\alpha(T_B)$ with $1 < p < \infty$. Then $f_j \to 0$ weakly in $A^p_\alpha(T_B)$ as $j \to \infty$ if and only if $\{f_j\}$ is bounded in $A^p_\alpha(T_B)$ and converges to 0 uniformly on each compact subset of T_B.*

Proof. (1) We first prove the necessary part of the lemma. Suppose $\{f_j\}$ converges to 0 weakly in $A^p_\alpha(T_B)$ as $j \to \infty$, then $\{f_j\}$ is bounded in $A^p_\alpha(T_B)$ according to the uniform boundedness principle. Based on formula (2.1), $\{f_j\}$ is uniformly bounded on every compact subset of $A^p_\alpha(T_B)$ and thus is a normal family. Note that $\{f_j\}$ converges to 0 pointwise according to the property of the reproducing kernel as $j \to \infty$, so $\{f_j\}$ converges uniformly to 0 on every compact subset of T_B as $j \to \infty$.

(2) Now we prove the sufficient part of the lemma. Suppose $\sup_j \|f_j\|_{p,\alpha} < \infty$ and $\{f_j\}$ converges uniformly to 0 on every compact subset of T_B as $j \to \infty$. For any $\varepsilon > 0$ and any $g \in A^q_\alpha(T_B)$ (where $\frac{1}{p} + \frac{1}{q} = 1$), there exists a compact subset K of T_B such that

$$\Big(\int\limits_{T_B \setminus K} |g(z)|^q \rho(z)^\alpha \, dV(z) \Big)^{\frac{1}{q}} < \frac{\varepsilon}{2M}.$$

Then through the Hölder inequality, we have

$$\begin{aligned} |(f_j, g)| &\le \left| \int\limits_K f_j(z) \overline{g(z)} \rho(z)^\alpha \, dV(z) \right| + \left| \int\limits_{T_B \setminus K} f_j(z) \overline{g(z)} \rho(z)^\alpha \, dV(z) \right| \\ &\le V(k)^{\frac{1}{p}} \|g(z)\|_{q,\alpha} \sup_{z \in K} |f_j(z)| + \|f_j(z)\|_{p,\alpha} \Big(\int\limits_{T_B \setminus K} |g(z)|^q \rho(z)^\alpha \, dV(z) \Big)^{\frac{1}{q}}. \end{aligned}$$

Since $\{f_j\}$ uniformly converges to 0 on K as $j \to \infty$, there exists $N > 0$ such that when $j \ge N$, the second term of the above formula is less than $\frac{\varepsilon}{2}$. Therefore, $(f_j, g) \to 0$ holds when $j \to \infty$, and then $\{f_j\}$ converges weakly to 0 in $A^p_\alpha(T_B)$ as $j \to \infty$. □

Lemma 3.9. *As $|z|$ approaches infinity, $\rho(z, \mathbf{i})$ also tends to infinity.*

Proof. We consider two cases.

Case 1: When z_n is fixed and $|z_k| \to \infty$ for $k = 1, 2, \ldots, n-1$, we have

$$\begin{aligned} 4|\rho(z, \mathbf{i})| &\ge 4 \operatorname{Re} \rho(z, \mathbf{i}) \\ &= \sum_{k=1}^{n-1} \operatorname{Re} z_k^2 + 2 \operatorname{Im} z_n + 2 \\ &\ge \sum_{k=1}^{n-1} \operatorname{Re} z_k^2 + 2 \sum_{k=1}^{n-1} (\operatorname{Im} z_k)^2 + 2 \\ &= \sum_{k=1}^{n-1} (\operatorname{Re} z_k)^2 + \sum_{k=1}^{n-1} (\operatorname{Im} z_k)^2 + 2 \\ &= \sum_{k=1}^{n-1} |z_k|^2 + 2. \end{aligned}$$

This implies $|\rho(z, \mathbf{i})| \to \infty$ as $|z_k| \to \infty$ for $k = 1, 2, \ldots, n-1$.

Case 2: When the previous $n-1$ variables are fixed (assuming they are all zero, without loss of generality), then

$$2|\rho(z, \mathbf{i})| \ge |z_n| - 1,$$

which shows that $|\rho(z, \mathbf{i})| \to \infty$ as $|z_n| \to \infty$.

Therefore, from Cases 1 and 2, we can obtain that $|\rho(z, \mathbf{i})| \to \infty$ as $|z| \to \infty$, which completes the proof. □

In the Bergman spaces on the unit disk, it is known that as $|z|$ approaches the boundary, the normalized reproducing kernel weakly converges to 0 in $A^p_\alpha(T_B)$, as referenced in [20]. The following lemma informs us that in the Bergman spaces on tubes, the normalized reproducing kernel also possesses this property.

Lemma 3.10. *For $1 < p < \infty$ and $\alpha > -1$, we have $K_{\alpha,z} \|K_{\alpha,z}\|^{-1}_{p,\alpha} \to 0$ weakly in $A^p_\alpha(T_B)$ as $z \to \partial \widehat{T_B}$.*

Proof. According to Lemma 3.8, we have to prove uniform convergence to 0 of $K_{\alpha,z}\|K_{\alpha,z}\|_{p,\alpha}^{-1}$ on each $Q_j := \overline{D(\mathbf{i}, j)}$.

Applying Lemma 3.7 and Lemma 3.2, we can establish the existence of a constant $C > 0$ such that

$$\sup_{w\in Q_j}\left|\frac{K_{\alpha,z}(w)}{\|K_{\alpha,z}\|_{p,\alpha}}\right| \le C\frac{\rho(z)^{\frac{n+\alpha+1}{p'}}}{|\rho(z,\mathbf{i})|^{n+\alpha+1}}$$

for all $z \in T_B$. Given that $2\rho(z,\mathbf{i}) = \frac{1}{2}|z'^2 - 2iz_n + 2| \ge 1$ for all $z \in T_B$, we deduce

$$\sup_{w\in Q_j}\left|\frac{K_{\alpha,z}(w)}{\|K_{\alpha,z}\|_{p,\alpha}}\right| \le C\rho(z)^{\frac{n+\alpha+1}{p'}},$$

which indicates that $K_{\alpha,z}\|K_{\alpha,z}\|_{p,\alpha}^{-1} \to 0$ uniformly on Q_j as $z \to \partial T_B$.

Furthermore, through the utilization of Lemma 3.6 and Lemma 3.9, we ascertain that

$$\sup_{w\in Q_j}\left|\frac{K_{\alpha,z}(w)}{\|K_{\alpha,z}\|_{p,\alpha}}\right| \le \frac{C}{|\rho(z,\mathbf{i})|^{\frac{n+\alpha+1}{p}}} \to 0,$$

which demonstrates that $K_{\alpha,z}\|K_{\alpha,z}\|_{p,\alpha}^{-1} \to 0$ uniformly on Q_j as $|z| \to \infty$. □

4 Carleson measures on T_B

As mentioned in the introduction, the importance of Carleson measures in studying the analytical properties of Toeplitz operators is significant. In the following, we will introduce two important theorems related to Carleson measures, which provide powerful support for the proof of the main results, Theorems 6.1 and 6.2.

We first review the definition of Carleson measure. Let μ be a positive Borel measure on T_B and $p > 0$. We say that μ is Carleson measure for the Bergman Space $A_\alpha^p(T_B)$ if there exists a positive constant $C > 0$ such that

$$\int_{T_B} |f(z)|^p \, d\mu(z) \le C\|f\|_{p,\alpha}^p$$

for every $f \in A_\alpha^p(T_B)$.

A positive Borel measure μ is a vanishing Carleson measure if for any bounded sequence f_k in $A_\alpha^p(T_B)$ that converges uniformly to 0 on every compact subset of T_B, we have

$$\lim_{k\to\infty}\int_{T_B} |f_k|^p \, d\mu = 0.$$

The following theorem presents some equivalent characterizations of Carleson measures on tubes, and this theorem is derived from [15].

Theorem 4.1 ([15]). *Suppose $0 < p < \infty$, $r > 0$, $\alpha > -1$ and μ is a positive Borel measure on T_B. Then the following conditions are equivalent:*

(1) *μ is a Carleson measure for $A_\alpha^p(T_B)$.*

(2) *There exists a constant $C > 0$ such that*

$$\int_{T_B} \frac{\rho(z)^{n+\alpha+1}}{|\rho(z,w)|^{2(n+\alpha+1)}} \, d\mu(w) \le C$$

for all $z \in T_B$.

(3) *There exists a constant $C > 0$ such that*

$$\mu(D(z,r)) \le C\rho(z)^{n+\alpha+1}$$

for all $z \in T_B$.

(4) *There exists a constant $C > 0$ such that*

$$\mu(D(a_k,r)) \le C\rho(a_k)^{n+\alpha+1}$$

for all $k \ge 1$, where $\{a_k\}$ is an r-lattice in the Bergman metric.

The proof of Theorem 4.1 has been provided in [15], and we will not repeat it here.

In the next theorem, we provide some equivalent characterizations of vanishing Carleson measures. These characterizations illustrate the property of being a vanishing Carleson measure for $A^p_\alpha(T_B)$, which depends neither on p nor on r.

Theorem 4.2. *Suppose $1 < p < \infty$, $r > 0$, $\alpha > -1$ and μ is a positive Borel measure on T_B. Then the following conditions are equivalent:*

(1) *μ is a vanishing Carleson measure for $A^p_\alpha(T_B)$.*

(2) *The measure μ satisfies*

$$\lim_{a\to\partial\widehat{T_B}} \int_{T_B} \frac{\rho(a)^{n+\alpha+1}}{|\rho(z,a)|^{2(n+\alpha+1)}}\, d\mu(z) = 0.$$

(3) *The measure μ has the property that*

$$\lim_{a\to\partial\widehat{T_B}} \frac{\mu(D(a,r))}{\rho(a)^{n+\alpha+1}} = 0.$$

(4) *For an r-lattice $\{a_k\}$ in the Bergman metric, we have*

$$\lim_{k\to\infty} \frac{\mu(D(a_k,r))}{\rho(a_k)^{n+\alpha+1}} = 0.$$

Proof. (1) $\Rightarrow$ (2) According to Lemmas 3.7 and 3.10, take

$$g_a(z) = \left(\frac{\rho(a)^{n+\alpha+1}}{|\rho(z,a)|^{2(n+\alpha+1)}}\right)^{\frac{1}{p}}.$$

By Lemmas 3.7 and 3.10, we see that $g_a(z)$ converges weakly to 0 in $A^p_\alpha(T_B)$ as $a \to \partial\widehat{T_B}$. Therefore,

$$\int_{T_B} \frac{\rho(a)^{n+\alpha+1}}{|\rho(z,a)|^{2(n+\alpha+1)}}\, d\mu(z) = C\int_{T_B} |g_a(z)|^p\, d\mu(z) \to 0$$

as $a \to \partial\widehat{T_B}$, which shows that (2) holds.

(2) $\Rightarrow$ (3) Obviously, the following equation holds:

$$\lim_{a\to\partial\widehat{T_B}} \int_{T_B} \frac{\rho(a)^{n+\alpha+1}}{|\rho(z,a)|^{2(n+\alpha+1)}}\, d\mu(z) = 0.$$

By Lemma 3.2, $|\rho(z,a)|$ and $\rho(z)$ are comparable when $z \in D(a,r)$. Therefore, μ has that property.

(3) $\Rightarrow$ (4) We know that $a_k \to \partial\widehat{T_B}$ as $k \to \infty$ if $\{a_k\}$ is an r-lattice in the Bergman metric. So the conclusion is trivial.

(4) $\Rightarrow$ (1) If the equality holds, we show that the inclusion map i_p from $A^p_\alpha(T_B)$ into $L^p(T_B, d\mu)$ is compact. To this end, we assume that $\{f_j\}$ is a sequence in $A^p_\alpha(T_B)$ that converges to 0 uniformly on compact subsets of T_B and $\|f_j\|_{p,\alpha} \le M$ for some positive constant M. By assumption, given $\varepsilon > 0$, there exists a positive integer N_0 such that

$$\frac{\mu(D(a_k,r))}{\rho(a_k)^{n+\alpha+1}} < \varepsilon, \quad k \ge N_0.$$

By Lemma 3.1, there is a constant $C > 0$ such that

$$\begin{aligned}\sum_{k=N_0}^{\infty} \int_{D(a_k,r)} |f_j(z)|^p\, d\mu(z) &\le C\sum_{k=N_0}^{\infty} \frac{\mu(D(a_k,r))}{\rho(a_k)^{n+\alpha+1}} \int_{D(a_k,2r)} |f_j(z)|^p\, dV_\alpha(z)\\ &\le \varepsilon CN\int_{T_B} |f_j(z)|^p\, dV_\alpha(z) \le \varepsilon CNM^p\end{aligned}$$

for all j, where C, N and M are all independent of ε. Since

$$\lim_{j\to\infty} \sum_{k=1}^{N_0-1} \int_{D(a_k,r)} |f_j(z)|^p\, d\mu(z) = 0,$$

we have

$$\lim_{j\to\infty}\sup\int_{T_B}|f_j(z)|^p d\mu(z) \le \lim_{j\to\infty}\sup\left[\sum_{k=1}^{N_0-1}\int_{D(a_k,r)}|f_j(z)|^p\,d\mu(z)+\sum_{k=N_0}^{\infty}\int_{D(a_k,r)}|f_j(z)|^p\,d\mu(z)\right]$$
$$\le \varepsilon CNM^p.$$

Since ε is arbitrary, we find that μ is a vanishing Carleson measure. □

5 Dense subspaces of $A^p_\alpha(T_B)$

For the case of unit disk $\mathbb{D}$, we know that the space $H^\infty(\mathbb{D})$ of bounded analytic functions is dense in the Bergman spaces $A^p_\alpha(\mathbb{D})$. Since the Toeplitz operator is an integral operator, it is always well-defined, making the Toeplitz operator densely defined in $A^p_\alpha(\mathbb{D})$.

However, in the case of tubes, there is no dense property as good as in a unit disk. In order to study the theory of Toeplitz operators on the Bergman space $A^p_\alpha(T_B)$ over tubular domains T_B, we have first to find a dense subspaces of $A^p_\alpha(T_B)$, and then obtain the Toeplitz operators densely defined on $A^p_\alpha(T_B)$.

To do this, we first introduce the following definition: Let $\mathcal{M}_+$ be the set of all positive Borel measures μ such that

$$\int_{T_B}\frac{d\mu(z)}{|\rho(z,\mathbf{i})|^t}<\infty$$

for some $t>0$.

Given $\mu\in\mathcal{M}_+$, the Toeplitz operator T_μ with symbol μ is given by

$$T_\mu f(z) := \int_{T_B} K_\alpha(z,w)f(w)\,d\mu(w).$$

Given $t\in\mathbb{R}$, we denote by S_t the vector spaces of functions f holomorphic in T_B satisfying

$$sup_{z\in T_B}|\rho(z,\mathbf{i})|^t|f(z)|<\infty.$$

Theorem 5.1. *If $1\le p<\infty$, $\alpha>-1$, $t>n+\frac{\alpha+1}{p}$ and $\mu\in\mathcal{M}_+$, then the Toeplitz operator T_μ is densely defined on $A^p_\alpha(T_B)$.*

Proof. To prove this theorem, we need to find a dense subspace S_t of $A^p_\alpha(T_B)$ such that for any $f\in S_t$, the following equation holds:

$$\int_{T_B}|K_\alpha(z,w)f(w)|\,d\mu(w)<\infty.$$

Now, we prove the density of S_t in $A^p_\alpha(T_B)$. Let $f_j=f\cdot\chi_{Q_j}$ for $j=1,2,\ldots$, where $Q_j=\overline{D(\mathbf{i},j)}$ and χ_{Q_j} is the characteristic function of Q_j. Clearly, $\|f_j-f\|_{p,\alpha}\to 0$ as $j\to\infty$. Given $\lambda>-1$, let P_λ be the integral operator given by

$$P_\lambda g(z)=c_\lambda\int_{T_B}\frac{\rho(w)^\lambda}{\rho(z,w)^{n+1+\lambda}}g(w)\,dV(w),\quad z\in T_B,$$

where

$$c_\lambda=\frac{2^{n+1+2\lambda}\Gamma(n+1+\lambda)}{\pi^n\Gamma(1+\lambda)}.$$

It was shown in [9, Theorem 3.2] that P_λ is a bounded projection from $L^p_\alpha(T_B)$ onto $A^p_\alpha(T_B)$, provided that $\lambda>\frac{\alpha+1}{p}-1$. Taking $\lambda=t-n-1$, by Hölder's inequality, we obtain

$$|P_{t-n-1}f_j(z)|\le c_{t-n-1}\|f\|_{p,\alpha}V_\alpha(Q_j)^{\frac{1}{p'}}\sup_{w\in Q_j}\frac{\rho(w)^{t-n-\alpha-1}}{|\rho(z,w)|^t}$$

for all $z\in T_B$. Combined with Lemma 3.3, we only need to consider the upper bound of

$$\sup_{w\in Q_j}\frac{\rho(w)^{t-n-\alpha-1}}{|\rho(z,w)|^t}.$$

Since Q_j are fixed, a simple calculation yields

$$|P_{t-n-1}f_j(z)| \le \frac{C\|f\|_{p,\alpha}}{|\rho(z,\mathbf{i})|^t}$$

for all $z \in T_B$, where $C > 0$ is a constant independent of z. Thus

$$P_{t-n-1}f_j(z) \in S_t.$$

Since $f \in A^p_\alpha(T_B)$ and $P_{\alpha-n-1}$ is a bounded projection from $L^p_\alpha(T_B)$ onto $A^p_\alpha(T_B)$, we have

$$\|P_{\alpha-n-1}f_j(z) - f\|_{p,\alpha} = \|P_{\alpha-n-1}(f_j(z) - f)\|_{p,\alpha} \le \|P_{\alpha-n-1}\|\|f_j - f\|_{p,\alpha} \to 0$$

as $j \to \infty$. This implies that S_t is dense in $A^p_\alpha(T_B)$.

According to Lemma 3.6, we know that there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{T_B} |K_\alpha(z,w)f(w)|d\mu(w) &\lesssim \frac{1}{\rho(z)^{n+\alpha+1}} \int_{T_B} \frac{|\rho(w,\mathbf{i})|^t|f(w)|}{|\rho(w,\mathbf{i})|^t}\, d\mu(w)\\ &\lesssim \frac{1}{\rho(z)^{n+\alpha+1}} \int_{T_B} \frac{d\mu(w)}{|\rho(w,\mathbf{i})|^t} < \infty\end{aligned}$$

holds. This completes the proof of the theorem. □

6 Characterization of Toeplitz operators

Based on the previously obtained fundamental results, we can now prove the following result.

Theorem 6.1. *Suppose that $r > 0$, $1 < p < \infty$, $0 < q < \infty$, $\alpha > -1$ and $\mu \in \mathcal{M}_+$. Then the following conditions are equivalent:*

(1) *T_μ is bounded on $A^p_\alpha(T_B)$.*
(2) *$\tilde{\mu}$ is a bounded function on T_B.*
(3) *$\hat{\mu}_r$ is a bounded function on T_B.*
(4) *μ is a Carleson measure for $A^q_\alpha(T_B)$.*

Proof. (1) ⇒ (2) We have the well-known conclusion

$$\langle T_\mu k_z, k_z\rangle = \int_{T_B} |k_z(w)|^2\, d\mu(w) = \tilde{\mu}(z),$$

and, according to the conditions and Lemma 3.7, we have

$$|\langle T_\mu k_z, k_z\rangle| \le \|T_\mu k_z\|_{p,\alpha}\|k_z\|_{p',\alpha} \le \frac{\|T_\mu\|\|K_z\|_{p,\alpha}\|K_z\|_{p',\alpha}}{K(z,z)} = C\|T_\mu\|.$$

Therefore, $\tilde{\mu}(z)$ is a bounded function on T_B.

(2) ⇒ (3) By the definition of $\hat{\mu}_r$ and Theorem 4.1, the expected results are obtained by the following identity:

$$V_\alpha(D(z,r)) = K\rho(z)^{n+\alpha+1},$$

where K is a constant independent of z.

(3) ⇒ (4) This result can be obtained from the fact that the Carleson measure in Theorem 4.1 is not dependent on p, so it is trivial.

(4) ⇒ (1) Since μ is the Carlson measure for $A^q_\alpha(T_B)$, and since the Carlson measure does not depend on the index, we know that μ is the Carlson measure for $A^1_\alpha(T_B)$. Combining this with fact that $T_\mu f$ is well defined on S_t, we conclude that there exists a constant $C > 0$ such that

$$\int_{T_B} |K_\alpha(z,w)f(w)|\, d\mu(w) \le C\int_{T_B} |K_\alpha(z,w)f(w)|\, dV_\alpha(w) \le C\int_{T_B} \frac{\rho(w)^\alpha\, dV(w)}{|\rho(z,w)|^{n+\alpha+1}|\rho(w,\mathbf{i})|^t}.$$

According to the holomorphic automorphism from $\mathbb{B}_n$ to T_B, let

$$w = \Phi(\xi), \quad z = \Phi(\eta),$$

and make a variable transformation of the above integral. By Lemma 3.5, we obtain

$$\int_{T_B} \frac{\rho(w)^\alpha \, dV(w)}{|\rho(z,w)|^{n+\alpha+1}|\rho(w,\mathbf{i})|^t} = 2^{n+1}|1+\eta_n|^{n+\alpha+1} \int_{\mathbb{B}_n} \frac{(1-|\xi|^2)^\alpha \, dV(\xi)}{|1-\langle \eta, \xi\rangle|^{n+\alpha+1}|1+\xi_n|^{n+\alpha+1-t}}.$$

By [18, Theorem 3.1], the last integral is dominated by a constant multiple of

$$\frac{1}{|1+\eta_n|^{n+\alpha+1-t}}\left(1+\log\frac{|1+\eta_n|}{1-|\eta|^2}\right).$$

Thus, there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{T_B} \frac{\rho(u)^\alpha \, dV(u)}{|\rho(z,u)|^{n+\alpha+1}|\rho(u,\mathbf{i})|^t} &\le C|1+\eta_n|^t\left(1+\log\frac{1}{1-|\eta|^2}\right)\\ &= \frac{C}{|\rho(z,\mathbf{i})|^t}\left(1+\log\frac{|\rho(z,\mathbf{i})|^2}{\rho(z)}\right)\end{aligned}$$

holds. By the elementary inequality $\log x < x^\varepsilon$, where

$$x > 1, \quad 0 < \varepsilon < \min\left\{\frac{\alpha+1}{p}, t - \frac{\alpha+n+1}{p}\right\},$$

we have

$$\int_{T_B} |T_\mu f(z)|^p \, dV_\alpha(z) \le C\left(\int_{T_B} \frac{dV_\alpha(z)}{|\rho(z,\mathbf{i})|^{pt}} + \int_{T_B} \frac{\rho(z)^{-p\varepsilon}}{|\rho(z,\mathbf{i})|^{p(t-2\varepsilon)}} \, dV_\alpha(z)\right).$$

This shows that T_μ is densely defined on S_t and can be extended to be a bounded operator on $A^p_\alpha(T_B)$. □

The above Theorem 6.1 characterizes the boundedness of the Toeplitz operator T_μ. Now, we will characterize the compactness of T_μ in the following theorem.

Theorem 6.2. *Suppose that $r > 0, 1 < p, q < \infty, \alpha > -1$ and $\mu \in \mathcal{M}_+$. Then the following conditions are equivalent:*
(1) *T_μ is compact on $A^p_\alpha(T_B)$.*
(2) *$\tilde{\mu}$ belongs to $C_0(T_B)$.*
(3) *$\hat{\mu}_r$ belongs to $C_0(T_B)$.*
(4) *μ is a vanishing Carleson measure for $A^q_\alpha(T_B)$.*

Proof. (1) ⇒ (2) Assume that T_μ is a compact operator on $A^p_\alpha(T_B)$ for some $p > 1$. Since

$$|\tilde{\mu}(z)| \le C\|T_\mu\left(\frac{K_z}{\|K_z\|_{p,\alpha}}\right)\|_{p,\alpha}$$

for all $z \in T_B$, combining the compactness of T_μ and Lemma 3.10, we obtain $\tilde{\mu} \in C_0(T_B)$.

(2) ⇒ (3) By the definition of $\hat{\mu}_r$ and Theorem 4.2, the expected results are obtained by the following identity:

$$V_\alpha(D(z,r)) = K\rho(z)^{n+\alpha+1},$$

where K is a constant independent of z.

(3) ⇒ (4) This conclusion can be drawn from the fact that the vanishing Carleson measure in Theorem 4.2 is not dependent on p, so it is trivial.

(4) ⇒ (1) Since μ is a vanishing Carlson measure for $A^q_\alpha(T_B)$, and according to the fact that the vanishing Carlson measure does not depend on the index, we know μ is the vanishing Carleson measure for $A^{p'}_\alpha(T_B)$. By Theorem 6.1, we know that T_μ is bounded on $f \in A^p_\alpha(T_B)$. For any $f \in A^p_\alpha(T_B)$, we have

$$\begin{aligned}\|T_\mu f\|_{p,\alpha} &= \sup\{|\langle T_\mu f, g\rangle| : \|g\|_{p',\alpha} = 1 \text{ in } A^{p'}_\alpha(T_B)\}\\ &= \sup\left\{\left|\int_{T_B} f\overline{g} \, d\mu\right| : \|g\|_{p',\alpha} = 1 \text{ in } A^{p'}_\alpha(T_B)\right\}\\ &\le \|f\|_{L^p(\mu)} \sup\{\|g\|_{L^{p'}(\mu)} : \|g\|_{p',\alpha} = 1 \text{ in } A^{p'}_\alpha(T_B)\},\end{aligned}$$

where the rationality of the second equation uses the Fubini theorem. Since μ is a vanishing Carleson measure on $A^{p'}_{\alpha}(T_B)$, the second term of the last inequality is finite. Now, if $f_j \to 0$ weakly in $A^p_\alpha(T_B)$, then the compactness of the inclusion mapping implies that $\|f\|_{L^p(\mu)} \to 0$. It follows that $\|T_\mu f_j\|_{p,\alpha} \to 0$, and hence T_μ is compact. □

Acknowledgment: The authors are sincerely grateful to the referee for careful reading and valuable suggestions.

Funding: This work is supported by the National Natural Science Foundation of China (Grants No. 12326407 and No. 12071155).

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Complex Analysis and Operator Theory (2025) 19:46
https://doi.org/10.1007/s11785-025-01669-8

Complex Analysis and Operator Theory

BMO on Weighted Bergman Spaces Over Tubular Domains

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Received: 30 September 2024 / Accepted: 7 February 2025 / Published online: 27 February 2025
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Abstract
In this paper, we characterize Bounded Mean Oscillation (BMO) and establish their connection with Hankel operators on weighted Bergman spaces over tubular domains. By utilizing the space BMO, we provide a new characterization of Bloch spaces on tubular domains. Next, we define a modified projection operator and prove its boundedness. Furthermore, we introduce differential operators and demonstrate that these operators belong to Lebesgue spaces on tubular domains. Finally, we establish an integral representation for Bergman functions using these differential operators.

Keywords BMO · Bergman space · Bloch space · Differential operator · Tubular domain

1 Introduction

BMO plays a critical role in harmonic analysis and has been extensively studied, with significant contributions from researchers such as Garnett [1] and Fefferman-Stein [2]. Zhu [11] studied BMO in the unit ball, providing an equivalent characterization that

Communicated by Irene Sabadini.

The research of the second author was supported by NSFC (Grant No. 12326407, 12226318, and 1901205); the fourth author was supported by Beijing Natural Science Foundation (No. 1252005).

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has influenced later research. Békollé et al. [20] extended the investigation to BMO on bounded symmetric domains, while Pau et al. [13] considered the weighted BMO within the unit ball.

The Bloch space is characterized by its invariance under biholomorphic transformations and Möbius maps, highlighting its significance in complex analysis. And its properties have been studied by many authors ([3–5, 8–10, 15] and [17]), with extensions from the unit disk to the unit ball, and further to the unbounded domain of the Siegel upper half-plane. For the cases of the unit disk and unit ball, many foundational results have established equivalences with L^∞ spaces. Further studies in the Siegel upper half-plane have explored analogous characterizations, although these come with limitations for specific range of p. In this paper, we mainly consider another unbounded domains, such as tubular domain T_B. Following the approach taken in the cases of the unit ball and Siegel upper half-plane, we utilize invariant gradients to define the Bloch space $\mathcal{B}$ over T_B.

The main ideas of this paper are inspired by the works of Liu [19] and Zhu [11], with Zhu's work in particular investigating key results concerning BMO and Bloch space within the unit ball. In contrast, our work extends these results to tubular domains, providing a broader framework for understanding the relationship between these spaces and their applications. While Si [17] has explored similar results to the Siegel upper half-plane.

In this paper, we obtain four main results. First, we establish a characterization of the BMO_α^p on T_B by the bounded Hankel operators on weighted Bergman spaces. Utilizing BMO_α^p, we derive a new characterization of Bloch space. Additionally, we demonstrate that the Bergman kernel does not belong to $L^1_\alpha(T_B)$ within the tubular domains, thereby introducing a modified kernel function $\widetilde{K}_\alpha$ and a modified projection operator $\widetilde{P}_\alpha$. We also prove that the modified projection from $L^\infty(T_B)$ to $\widetilde{\mathcal{B}}$ is bounded and provide a kind of integral representation for Bergman functions.

To aid the readers' understanding and reference, we present the necessary symbols in the following table.

Symbol	Standard form of the symbol
$\widehat{f_r}(z)$	$\frac{1}{\lvert D(z,r)\rvert}\int_{D(z,r)} f(w)\,dV_\alpha(w)$
$MO_r(f)(z)$	$\Big(\frac{1}{\lvert D(z,r)\rvert}\int_{D(z,r)} \lvert f(w)-\widehat{f_r}(z)\rvert^p\,dV_\alpha(w)\Big)^{1/p}$
$\lVert f\rVert_{BMO_r^p}$	$\sup\{MO_r(f)(z) : z\in T_B\}$
$f\in VMO_r$	$MO_r(f)\in C_0(T_B)$
$\omega_r(f)(z)$	$\sup\{\lvert f(z)-f(w)\rvert : w\in D(z,r)\}$
$\lVert f\rVert_{BO_r}$	$\sup\{\omega_r(f)(z) : z\in T_B\}$
BA_r^p	$\widehat{\lvert f\rvert_r^p}(z)\in L^\infty(T_B)$

Theorem A *For $r>0$, $1\le p<\infty$, and $p(\alpha+1)>\lambda+1$, let $f\in L^p_\lambda(T_B)$, the following conditions are equivalent:*

(a) $f\in BMO_r^p$.

(b) $f = f_1 + f_2$ *with* $f_1 \in BO$ *and* $f_2 \in BA^p$.
(c) $H_f^{(\alpha)}$ *and* $H_{\overline{f}}^{(\alpha)}$ *are both bounded on* $A^p_\lambda(T_B)$.

Following the characterization of BMO_r^p in Theorem A, we now investigate the relationship between BMO and the Bloch spaces defined on tubular domains.

Theorem B *Let* $H(T_B)$ *be the space of holomorphic functions in* T_B. *For any* $r > 0$, *we have*

$$\mathcal{B} = H(T_B) \cap BMO_r^p,$$

and

$$\mathcal{B}_0 = H(T_B) \cap VMO_r^p.$$

Moreover, the two semi-norms $\|f\|_{\mathcal{B}}$ *and* $\|f\|_{BMO_r^p}$ *are equivalent.*

Next, we will consider a new space

$$\widetilde{\mathcal{B}} := \{f \in \mathcal{B} : f(\mathbf{i}) = 0\},$$

where $\mathbf{i} := (0', i)$, and $0' = (0, \cdots, 0) \in \mathbb{R}^{n-1}$.

The Bloch space on the unit ball is known to be equivalent to $L^\infty(\mathbb{B}_n)$ when considered under the projection operator P_α. We aim to extend this relationship to tubular domains. However, due to Forreli-Rudin type estimates showing that the Bergman kernel does not belong to $L^1_\alpha(T_B)$ on tubular domains, it becomes necessary to define a modified kernel function, $\widetilde{K_\alpha}$, and a corresponding modified projection operator, $\widetilde{P_\alpha}$, from $L^\infty(T_B)$ to $\widetilde{\mathcal{B}}$, as described below:

$$\widetilde{K_\alpha} := K_\alpha(z, w) - K_\alpha(\mathbf{i}, w),$$
$$\widetilde{P_\alpha} f(z) := \int_{T_B} \widetilde{K_\alpha}(z, w) f(w) dV_\alpha(w),$$

where $\mathbf{i} := (0', i)$.

Theorem C $\widetilde{P_\alpha}$ *is a bounded projection from* $L^\infty(T_B)$ *to* $\widetilde{\mathcal{B}}$.
For $\alpha > -1$, *we define the integral operator* $\mathcal{T}_\alpha$ *as follows:*

$$\mathcal{T}_\alpha f(z) = \frac{\Gamma(n+1+\alpha)}{2^{n+1}\pi^n \Gamma(\alpha+1)} \int_{T_B} \frac{\boldsymbol{\rho}(w)^\alpha}{\boldsymbol{\rho}(z, w)^{n+1+\alpha}} f(w) dV(w),$$

where $dV(z)$ *denotes the Lebesgue measure on* $\mathbb{C}^n$.

Theorem D *Suppose that* $1 \le p < \infty$, $\lambda > -1$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$\begin{cases} \alpha > \frac{\lambda+1}{p} - 1, & 1 < p < \infty, \\ \alpha \ge \lambda, & p = 1. \end{cases}$$

If $f \in A^p_\lambda(T_B)$ *then*

$$f = \frac{(2i)^N \Gamma(1+\alpha)}{\Gamma(1+\alpha+N)} \mathcal{T}_\alpha \left(\boldsymbol{\rho}^N \mathcal{L}^N_n f \right)$$

for any $N \in \mathbb{N}_0$.

The structure of this paper is organized as follows: In Sect. 2, we introduce key terminology and preliminary results. Section 3 is dedicated to the first two theorems, namely Theorem A and Theorem B. In Sect. 4, we focus on establishing the last two theorems, Theorem C and Theorem D.

Additionally, the notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$, and the notation $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2 Preliminaries and Auxiliary Results

2.1 Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For any two points $z = (z_1, \cdots, z_n)$ and $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_n)$ in $\mathbb{C}^n$, we write

$$\begin{aligned} z \cdot \bar{w} &:= z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n, \\ |z'|^2 &:= |z_1|^2 + |z_2|^2 + \cdots + |z_{n-1}|^2, \end{aligned}$$

where $z' = (z_1, \cdots, z_{n-1})$.

The set

$$T_B = \left\{ z = x + iy, x \in \mathbb{R}^n, y \in B \subset \mathbb{R}^n \right\}$$

is a tubular domain in an n-dimensional complex space $\mathbb{C}^n$, where

$$B := \left\{ y = (y', y_n) : \left|y'\right|^2 < y_n, y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbb{R}^{n-1} \right\}.$$

We define the space $L^p_\alpha(T_B)$, which consists of all Lebesgue measurable functions f on T_B, with the norm given by:

$$\|f\|_{L^p_\alpha(T_B)} = \left\{ \int_{T_B} |f(z)|^p \, dV_\alpha(z) \right\}^{\frac{1}{p}} < \infty,$$

where $dV_\alpha(z) = \left(y_n - \left|y'\right|^2\right)^\alpha dV(z)$, $\alpha > -1$, and $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$.

Let $A^p_\alpha(T_B)$ denote the weighted Bergman space on T_B. Since the evaluation functional is bounded, so $A^p_\alpha(T_B)$ is the closed subspace of $L^p_\alpha(T_B)$. It is easy to know

that, when $1 \leqslant p < \infty$, the space $A_\alpha^p(T_B)$ is a Banach space with the norm $\|\cdot\|_p$. Especially, when $p = 2$, $A_\alpha^2(T_B)$ is a Hilbert space.

The orthogonal projection from $L_\alpha^2(T_B)$ to $A_\alpha^2(T_B)$ is the following integral operator:

$$P_\alpha f(z) = \int_{T_B} K_\alpha(z, w) f(w) \, dV_\alpha(w),$$

with the Bergman kernel ([7])

$$K_\alpha(z, w) = \frac{2^{n+1+2\alpha} \Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\pi^n} \left(\left(z' - \overline{w'} \right)^2 - 2i (z_n - \overline{w}_n) \right)^{-n-1-\alpha}.$$

For convenience, we introduce the following notation:

$$\boldsymbol{\rho}(z, w) = \frac{1}{4} \left(\left(z' - \overline{w'} \right)^2 - 2i (z_n - \overline{w_n}) \right)$$

and let $\boldsymbol{\rho}(z) := \boldsymbol{\rho}(z, z) = y_n - |y'|^2$. Then the Bergman kernel of T_B is

$$K_\alpha(z, w) = \frac{\Gamma(n+1+\alpha)}{2^{n+1}\pi^n \Gamma(\alpha+1) \boldsymbol{\rho}(z, w)^{n+1+\alpha}}.$$

Moreover, for $\alpha, \lambda > -1$, let $1 \leq p < \infty$, P_α can be extend to $L_\lambda^p(T_B)$, and P_α is a bounded projection from $L_\lambda^p(T_B)$ onto $A_\lambda^p(T_B)$ if and only if $p(\alpha+1) > \lambda + 1$, see [16, Lemma 3.4].

For $f \in L_\alpha^p(T_B)$, let M_f denote the multiplication operator induced f, the Hankel operator with symbol f is denote by

$$H_f^\alpha = (\mathbf{I} - P_\alpha) M_f P_\alpha,$$

where $\mathbf{I}$ is the identity operator.

Let $bT_B := \{z \in \mathbb{C}^n : \boldsymbol{\rho}(z) = 0\}$ denote the boundary of T_B. Then $\widehat{T_B} := T_B \cup bT_B \cup \{\infty\}$ is the one-point compactification of T_B. Also, let $\partial \widehat{T_B} := bT_B \cup \{\infty\}$. Thus, $z \to \partial \widehat{T_B}$ means $\boldsymbol{\rho}(z) \to 0$ or $|z| \to \infty$.

Let $\mathcal{M}_+$ be the set of all positive Borel measure μ such that

$$\int_{T_B} \frac{d\mu(z)}{|\boldsymbol{\rho}(z, \mathbf{i})|^t} < \infty,$$

for some $t > 0$.

And let the complex matrix

$$\mathbf{B}(z) := (b_{ij}(z))_{1 \leq i, j \leq n} = \frac{1}{n+1+\alpha} \left(\frac{\partial^2}{\partial \bar{z}_i \partial z_j} \ln K_\alpha(z, z) \right)_{1 \leq i, j \leq n}$$

be the Bergman matrix of T_B. For a C^1 curve $\gamma : [0, 1] \to T_B$, we define

$$l(\gamma) = \int_0^1 \left\langle \mathbf{B}(\gamma(t))\gamma'(t), \gamma'(t) \right\rangle dt,$$

and call β the Bergman metric on T_B, where

$$\beta(z, w) = \inf\{l(\gamma) : \gamma(0) = 1, \gamma(1) = w\}.$$

Let $D(z, r)$ denote the Bergman metric ball at z with radius r. Thus

$$D(z, r) = \{w \in T_B : \beta(z, w) < r\}.$$

Let $|D(z, r)| = V_\alpha(D(z, r))$. For a locally integrable function f on T_B, we define a function $\widehat{f_r}$ on T_B as follows:

$$\widehat{f_r}(z) = \frac{1}{|D(z, r)|} \int_{D(z,r)} f(w) dV_\alpha(w),$$

and $\widehat{f_r}(z)$ is the integral mean of f over $D(z, r)$. For fixed $r > 0$ and $f \in L^p_\alpha(T_B)$, $1 \le p < \infty$, we define the mean oscillation of f at z in the Bergman metric as follows:

$$MO_r(f)(z) = \left(\frac{1}{|D(z, r)|} \int_{D(z,r)} |f(w) - \widehat{f_r}(z)|^p dV_\alpha(w) \right)^{1/p}.$$

The space BMO^p_r consists of those functions $f \in L^p_\alpha(T_B)$ such that

$$\|f\|_{BMO^p_r} = \sup\{MO_r(f)(z) : z \in T_B\} < \infty.$$

It is evident that the expression above defines a semi-norm, and to convert it into a norm, it suffices to add the constant $|f(0)|$.

We denote by $C_0(T_B)$ the space of complex-valued continuous functions f on T_B such that $f(z) \to 0$ as $z \to \partial\widehat{T_B}$. We say that $f \in VMO_r$ if $MO_r(f) \in C_0(T_B)$.

For a continuous function f on T_B, any $r > 0$, let

$$\omega_r(f)(z) = \sup\{|f(z) - f(w)| : w \in D(z, r)\}.$$

The function $\omega_r(f)(z)$ is called the oscillation of f at z in the Bergman metric. Let BO_r denote the space of continuous functions f for which the semi-norm is finite, that is,

$$\|f\|_{BO_r} = \sup\{\omega_r(f)(z) : z \in T_B\} < \infty.$$

Recall that the Berezin transform over T_B is denoted by

$$B_\alpha f(z) = \int_{T_B} f(w)|k_z^\alpha(w)|^2 dV_\alpha(w), \quad z \in T_B,$$

where

$$k_z^\alpha(w) = K_\alpha(w,z)/\sqrt{K_\alpha(z,z)}, \quad w \in T_B.$$

Actually, the Berezin transform is bounded on $L^p_\alpha(T_B)$ for $1 \le p < \infty$, see [7].

Let BA_r^p denote the spaces of all functions on T_B with the property that $\widehat{|f|_r^p}(z) \in L^\infty(T_B)$. From [16, Theorem 6.1], we know that $\widehat{|f|_r^p}(z)$ is bounded if and only if $B_\alpha(|f|^p)(z)$ is bounded, so which means BA_r^p is independent of r, simply write BA^p for BA_r^p.

2.2 Cayley Transform and Automorphism

We will use the transform $\Phi : \mathbb{B}_n \to T_B$ as follows, see [7]

$$\Phi(z) = \left(\frac{\sqrt{2}z'}{1+z_n}, i\frac{1-z_n}{1+z_n} - i\frac{z'\cdot z'}{(1+z_n)^2} \right), \quad z \in \mathbb{B}_n. \tag{2.1}$$

And it is not hard to calculate that

$$\Phi^{-1}(w) = \left(\frac{2iw'}{i+w_n+\frac{i}{2}w'\cdot w'}, \frac{i-w_n-\frac{i}{2}w'\cdot w'}{i+w_n+\frac{i}{2}w'\cdot w'} \right), \quad w \in T_B.$$

For each fixed $z \in T_B$, we give the holomorphic automorphism of T_B as follows:

$$h_z(u) := \left(u' - z', u_n - \mathrm{Re}z_n - iu'\overline{z'} + \frac{i\left|z'\right|^2}{2} + \frac{i\overline{z'}\cdot\overline{z'}}{4} + \frac{iu'\cdot z'}{4} \right).$$

$$\delta_t(u) = (tu', t^2u_n),\ t > 0,\ u \in T_B.2$$

Obviously, $h_z(u)$ is a holomorphic automorphism of T_B. Hence, the mapping $\sigma_z := \delta_{\rho(z)^{-\frac{1}{2}}} \circ h_z$ is a holomorphic automorphism of T_B. Simple calculations show that $\sigma_z(z) = \mathbf{i} := (0', i)$ and

$$(J_{\mathbb{C}}\sigma_z)(u) = \boldsymbol{\rho}(z)^{-\frac{n+1}{2}}, \tag{2.2}$$

where $(J_{\mathbb{C}}\sigma_z)(u)$ stands for the complex Jacobian of σ_z at u.

Through the Cayley transform, a class of Möbius transformations of T_B induced by φ_ξ is given by

$$\tau_z := \Phi \circ \varphi_{\Phi^{-1}(z)} \circ \Phi^{-1}. \tag{2.3}$$

Obviously $\Phi(0) = \mathbf{i}$, $\tau_z(z) = \mathbf{i}$.

To examine the properties of holomorphic functions on tubular domains, it is crucial to understand the behavior of holomorphic mappings from the unit ball to these domains. The following key lemma, adapted from [7], highlights essential properties of such mappings:

Lemma 2.1 *The following properties hold for holomorphic mapping Φ from $\mathbb{B}_n$ to T_B:*

(i) The real Jacobian of Φ at $z \in T_B$ is

$$J_R(\Phi(z)) = \frac{2^{n+1}}{|1 + z_n|^{2(n+1)}}. \tag{2.4}$$

(ii) The real Jacobian of Φ^{-1} at $z \in T_B$ is

$$(J_R\Phi^{-1})(z) = \frac{1}{4\,|\boldsymbol{\rho}(z, \mathbf{i})|^{2(n+1)}}. \tag{2.5}$$

(iii) The identity

$$1 - \left\langle \Phi^{-1}(z), \Phi^{-1}(w) \right\rangle = \frac{\boldsymbol{\rho}(z, w)}{\boldsymbol{\rho}(z, \mathbf{i})\boldsymbol{\rho}(\mathbf{i}, w)} \tag{2.6}$$

holds for all $z, w \in T_B$, where $\mathbf{i} = (0', i)$.
And moreover,

$$1 - \left|\Phi^{-1}(z)\right|^2 = \frac{\boldsymbol{\rho}(z)}{|\boldsymbol{\rho}(z, \mathbf{i})|^2}, 1 + [\Phi^{-1}(z)]_n = \frac{1}{\boldsymbol{\rho}(z, \mathbf{i})}.$$

(iv) The identity

$$\boldsymbol{\rho}(z, w) = \boldsymbol{\rho}(\Phi(\xi), \Phi(\eta)) = \frac{1 - \langle \xi, \eta \rangle}{(1 + \xi_n)(1 + \eta_n)} \tag{2.7}$$

holds for all $z, w \in T_B$, where $\xi = \Phi^{-1}(z)$, $\eta = \Phi^{-1}(w)$.

(v) For any $z \in T_B$, $\alpha > -1$ and $r > 0$ we have

$$V_\alpha\left(D\left(z, r\right)\right) \simeq \boldsymbol{\rho}\left(z\right)^{n+1+\alpha}. \tag{2.8}$$

2.3 The Bloch Space

In this section, we introduce the definition of the Bloch space on T_B. It is important to note that, while the Siegel upper half-space and the unit ball are biholomorphically equivalent, a clear and straightforward definition of the Bloch space on T_B has not been previously provided. To ensure that the Bloch space possesses Möbius invariance, we first introduce the invariant gradient.

For $z \in T_B$, let $(b_{i,j})$ denote an $n \times n$ Hermitian matrix and $(b^{i,j})$ its inverse. Set $b(z) = \det(b_{i,j}(z))$. Define

$$B(z) = \big(b_{i,j}(z)\big) = \frac{1}{n+1+\alpha}\left(\frac{\partial^2}{\partial\overline{z_i}\partial z_j}\ln K_\alpha(z,z)\right), 1 \le i, j \le n.$$

Let $Y = (y_1, y_2, \cdots, y_{n-1}, -\frac{1}{2})$, the matrix I_{n-1} is an $n-1$ dimensional identity matrix, and

$$A(y) = Y'Y, \quad I' = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\big(b_{i,j}(z)\big) = \frac{\frac{1}{2}(y_n - |y'|^2)I' + A(y)}{(y_n - |y'|^2)^2} = \frac{1}{\boldsymbol{\rho}(z)^2}\begin{pmatrix} \frac{\boldsymbol{\rho}(z)}{2}I_{n-1} + y'^T y' & -\frac{1}{2}y'^T \\ -\frac{1}{2}y' & \frac{1}{4} \end{pmatrix},$$

and we can get the inverse matrix

$$\left(b^{i,j}(z)\right) = \boldsymbol{\rho}(z)\begin{pmatrix} 2I_{n-1} & 4y'^T \\ 4y' & 4(y_n + |y'|^2) \end{pmatrix}, \ b(z) = \frac{1}{(2\boldsymbol{\rho}(z))^{n+1}}.$$

From [6], it is well known that the Laplace-Beltrami operator associated with the Bergman kernel K is the differential operator $\widetilde{\Delta}$ defined by

$$\widetilde{\Delta} = \frac{2}{b}\sum_{i,j}\left\{\frac{\partial}{\partial\bar{z}_i}\left(bb^{i,j}\frac{\partial}{\partial z_j}\right) + \frac{\partial}{\partial z_j}\left(bb^{i,j}\frac{\partial}{\partial\bar{z}_i}\right)\right\}.$$

Then substituting the above into the operator, we have

$$\begin{aligned}\widetilde{\Delta} = 8\boldsymbol{\rho}(z)\Bigg\{&\sum_{j=1}^{n-1}\frac{\partial^2}{\partial z_j\partial\bar{z}_j} + \sum_{j=1}^{n-1}2y_j\frac{\partial^2}{\partial z_n\partial\bar{z}_j} \\ &+ \sum_{j=1}^{n-1}2y_j\frac{\partial^2}{\partial\bar{z}_n\partial z_j} + 2(y_n + |y'|^2)\frac{\partial^2}{\partial\bar{z}_n\partial z_n}\Bigg\}\end{aligned} \tag{2.9}$$

The operator $\widetilde{\Delta}$ is often referred to the invariant Laplacian, see [14], since it has the following property.

Lemma 2.2

$$\widetilde{\Delta}(g \circ \psi) = (\widetilde{\Delta} g) \circ \psi \quad \textit{for all} \quad \psi \in \mathrm{Aut}(T_B),$$

where g is $C^2(T_B)$ and $\mathrm{Aut}(T_B)$ denote the group of automorphisms of T_B.

Proof From cayley transform, so we let

$$\Phi(z) = (w_1, w_2, \cdots, w_n) = \left(\frac{\sqrt{2} z'}{1 + z_n}, i \frac{1 - z_n}{1 + z_n} - i \frac{z' \cdot z'}{(1 + z_n)^2} \right), \quad z \in \mathbb{B}_n.$$

Though (2.3), let $\psi = \Phi \circ \varphi_{\Phi^{-1}(z)} \circ \Phi^{-1}$, and $\psi(z) = \mathbf{i}$. For the unit ball $\mathbb{B}_n$, the Laplace-Beltrami operator denote by $\widetilde{\Delta}_{\mathbb{B}_n}$, then from [6], we have

$$\widetilde{\Delta}_{\mathbb{B}_n} = 4(1 - |z|^2) \sum_{i,j=1}^{n} \left[\delta_{i,j} - \bar{z}_i z_j \right] \frac{\partial^2}{\partial z_j \partial \bar{z}_i},$$

where $\delta_{i,j}$ is the Kronecker delta. And

$$\widetilde{\Delta}_{\mathbb{B}_n}(f \circ \varphi) = (\widetilde{\Delta}_{\mathbb{B}_n} f) \circ \varphi,$$

where f is $C^2(\mathbb{B}_n)$, and $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

By Φ is holomorphic mapping, means $\frac{\partial w_i}{\partial \bar{z}_j} = 0$, then using the following complex version of the chain rule (see [21]):

$$\frac{\partial g}{\partial z_j} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial w_i} \frac{\partial w_i}{\partial z_j} + \frac{\partial f}{\partial \bar{w}_i} \frac{\partial \bar{w}_i}{\partial z_j} \right),$$
$$\frac{\partial g}{\partial \bar{z}_j} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial w_i} \frac{\partial w_i}{\partial \bar{z}_j} + \frac{\partial f}{\partial \bar{w}_i} \frac{\partial \bar{w}_i}{\partial \bar{z}_j} \right).$$

For any $j = 0, \cdots, n-1$, we have

$$\frac{\partial (f \circ \Phi)}{\partial \bar{z}_j} = \sqrt{2} \frac{\partial f(\mathbf{i})}{\partial \bar{w}_j},$$
$$\frac{\partial^2 (f \circ \Phi)(0)}{\partial z_j \partial \bar{z}_j} = 2 \frac{\partial^2 f(\mathbf{i})}{\partial w_j \partial \bar{w}_j},$$

and

$$\frac{\partial(f\circ\Phi)}{\partial\bar{z}_n}=2i\frac{\partial f(\mathbf{i})}{\partial\bar{w}_n},$$
$$\frac{\partial^2(f\circ\Phi)(0)}{\partial z_n\partial\bar{z}_n}=4\frac{\partial^2 f(\mathbf{i})}{\partial w_n\partial\bar{w}_n}.$$

Then we have

$$(\widetilde{\Delta}g)(\mathbf{i})=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi)(0).$$

Moreover, for $w\in\mathbb{B}_n$, we can choose $\varphi_w\in\mathrm{Aut}(\mathbb{B}_n)$, $\varphi_w(0)=w$, and $\Phi\circ\varphi_w$ is still a holomorphic mapping from $\mathbb{B}_n$ to T_B, so

$$\begin{aligned}\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi)(w)&=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi)(\varphi_w(0))\\&=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi\circ\varphi_w)(0)\\&=(\widetilde{\Delta}g)\circ(\Phi\circ\varphi_w(0))\\&=(\widetilde{\Delta}g)\circ(\Phi(w)).\end{aligned}$$

Hence, for $z\in T_B$, we have

$$\begin{aligned}(\widetilde{\Delta}g)\circ\psi(z)=(\widetilde{\Delta}g)(\mathbf{i})&=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi)(0)\\&=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi)\circ\varphi_{\Phi^{-1}(z)}\left(\Phi^{-1}(z)\right)\\&=\widetilde{\Delta}_{\mathbb{B}_n}(g\circ\Phi\circ\varphi_{\Phi^{-1}(z)})\circ\left(\Phi^{-1}(z)\right)\\&=\widetilde{\Delta}\left(g\circ\Phi\circ\varphi_{\Phi^{-1}(z)}\circ\Phi^{-1}\right)(z)\\&=\widetilde{\Delta}(g\circ\psi)(z).\end{aligned}$$

□

If u and v are $C^2(T_B)$ functions, then from [6],

$$\widetilde{\Delta}(uv)=u\widetilde{\Delta}v+2(\widetilde{\nabla}u)v+v\widetilde{\Delta}u,$$

where $\widetilde{\nabla}u$ is the vector field defined by

$$\widetilde{\nabla}u=2\sum_{i,j}b^{i,j}\left\{\frac{\partial u}{\partial\bar{z}_i}\frac{\partial}{\partial z_j}+\frac{\partial u}{\partial z_j}\frac{\partial}{\partial\bar{z}_i}\right\}.$$

Furthermore, if f in $H(T_B)$, it's not hard to check that

$$\widetilde{\Delta}|f|^2=2(\widetilde{\nabla}f)\overline{f}=2|\widetilde{\nabla}f|^2.$$

In particular, we have

$$|\widetilde{\nabla} f|^2 = 2\sum_{i,j} b^{i,j}\overline{\frac{\partial f}{\partial z_i}}\frac{\partial f}{\partial z_j}.$$

Hence, from the invariance property of $\widetilde{\Delta}$, we have

$$|\widetilde{\nabla}(f\circ\psi)| = |\widetilde{\nabla} f\circ\psi| \quad \text{for all} \quad \psi\in \mathrm{Aut}(T_B).$$

Therefore, the operator $|\widetilde{\nabla}|$ is usually called invariant gradient of T_B.

With this in mind, we will use the invariant gradient to directly define the Bloch space on T_B. The form of the invariant gradient is given by:

$$\left|\widetilde{\nabla} f(z)\right|^2 = 4\boldsymbol{\rho}(z)\left(2\boldsymbol{\rho}(z)\left|\frac{\partial f(z)}{\partial z_n}\right|^2 + \sum_{j=1}^{n-1}\left|\frac{\partial f(z)}{\partial z_j} + 2y_j\frac{\partial f(z)}{\partial z_n}\right|^2\right)$$

for $f\in H(T_B)$.

We define the Bloch space of T_B, denoted by $\mathcal{B}$, using invariant gradients. It is the space of functions $f\in H(T_B)$ such that

$$\|f\|_{\mathcal{B}} := \sup\{|\widetilde{\nabla} f(z)| : z\in T_B\} < \infty.$$

Clearly, this defines only a semi-norm, which is invariant under the action of $\mathrm{Aut}(T_B)$. The little Bloch space, denoted by $\mathcal{B}_0$, is defined as follows:

$$\mathcal{B}_0 := \left\{f\in\mathcal{B} : \widetilde{\nabla} f\in C_0\left(\widehat{T_B}\right)\right\},$$

where $C_0\left(\widehat{T_B}\right) = \{f\in C\left(\widehat{T_B}\right) : \lim_{z\to\partial\widehat{T_B}}|f(z)| = 0\}$.

Next, we define the differential operator as follows:

$$\mathcal{L}_n := \frac{\partial}{\partial z_n},\quad \mathcal{L}_j := \frac{\partial}{\partial z_j} + 2y_j\frac{\partial}{\partial z_n},\quad j = 1,\cdots,n-1.$$

For $\gamma\in\mathbb{N}_0^n$, which $\mathbb{N}_0$ denotes the set of non-negative integers, we write

$$\mathcal{L}^{\gamma} := (\mathcal{L}_1)^{\gamma_1}\cdots(\mathcal{L}_n)^{\gamma_n}.$$

The properties of the Bloch space will be established using the differential operator introduced earlier.

We will consider a new space

$$\widetilde{\mathcal{B}} := \{f\in\mathcal{B} : f(\mathbf{i}) = 0\},$$

where $\mathbf{i} := (0', i)$.

Using τ_z, we establish the relationship between the invariant gradients on T_B and $\mathbb{B}$ as follows. For any $f \in H(T_B)$, it is straightforward to verify that:

$$|\widetilde{\nabla} f(\mathbf{i})| = \sqrt{2}|\nabla(f \circ \Phi)(0)|, \tag{2.10}$$

then we have

$$\begin{aligned} |\widetilde{\nabla} f(z)| &= |\widetilde{\nabla}(f \circ \tau_z)(\mathbf{i})| = \sqrt{2}|\nabla(f \circ \tau_z \circ \Phi)(0)| \\ &= \sqrt{2}|\nabla(f \circ \Phi \circ \varphi_{\Phi^{-1}(z)})(0)| \\ &= \sqrt{2}|\widetilde{\nabla}_{\mathbb{B}_n}(f \circ \Phi)\big(\Phi^{-1}(z)\big)| \end{aligned} \tag{2.11}$$

for all $z \in T_B$.

2.4 Some Auxiliary Results

In order to establish our main theorems, we provide several auxiliary results that will support our proofs. We will omit most of the proofs for brevity.

Lemma 2.3 *([7, Therom 2]) There exists a positive integer N such that for any $0 < r \leqslant 1$ we can find a sequence $\{a_k\}$ in T_B with the following properites:*

(1) $T_B = \bigcup_{k=1}^{\infty} D(a_k, r)$;
(2) The sets $D(a_k, r/4)$ are mutually disjoint;
(3) Each point $z \in T_B$ belongs to at most N of the sets $D(a_k, r)$.

Lemma 2.4 *([7, Lemma 2]) Let $r, s > 0, t > -1$, and $r + s - t > n + 1$, then*

$$\int_{T_B} \frac{\boldsymbol{\rho}(w)^t}{\boldsymbol{\rho}(z, w)^r \boldsymbol{\rho}(w, u)^s} dV(w) = \frac{C_1(n, r, s, t)}{\boldsymbol{\rho}(z, u)^{r+s-t-n-1}} \tag{2.12}$$

for all $z, u \in T_B$, where

$$C_1(n, r, s, t) = \frac{2^{n+1}\pi^n \Gamma(1+t)\Gamma(r+s-t-n-1)}{\Gamma(r)\Gamma(s)}.$$

In particular, let $s, t \in \mathbb{R}$, if $t > -1, s - t > n + 1$, then

$$\int_{T_B} \frac{\boldsymbol{\rho}(w)^t}{|\boldsymbol{\rho}(z, w)|^s} dV(w) = \frac{C_1(n, s, t)}{\boldsymbol{\rho}(z)^{s-t-n-1}}.$$

Lemma 2.5 *([18, Theorem 2.2]) Suppose that $1 \leq p < \infty$, $\lambda > -1$ and $\alpha \in \mathbb{R}$ satisfies*

$$\begin{cases} \alpha > \frac{\lambda+1}{p} - 1, & 1 < p < \infty, \\ \alpha \geq \lambda, & p = 1. \end{cases}$$

If $f \in A^p_\lambda(T_B)$, then $f = \mathcal{T}_\alpha f$.

Lemma 2.6 *For any $r > 0$, the inequalities*

$$|\boldsymbol{\rho}(z,u)| \simeq |\boldsymbol{\rho}(z,v)|$$

hold for all $z, u, v \in T_B$ with $\beta(u,v) < r$.

Lemma 2.7 *([16, Lemma 3.7]) For any $z, w \in T_B$, we have*

$$2\,|\boldsymbol{\rho}(z,w)| \geq \max\{\boldsymbol{\rho}(z), \boldsymbol{\rho}(w)\}. \tag{2.13}$$

Lemma 2.8 *Let $1 \leq p < \infty$, $r > 0$, and $f \in L^p_\alpha(T_B)$, then $f \in BMO^p_r$ if and only if there is a constant λ_z such that, for all $z \in T_B$, with*

$$\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-\lambda_z|^p\, dV_\alpha(w) \leq C.$$

Proof If $f \in BMO^p_r$, then the inequality holds when $\lambda_z = \widehat{f_r}(z)$.

Conversely, assume that the above inequality holds for all $z \in T_B$, so by the triangle inequality, and Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \left|f(w)-\widehat{f_r}(z)\right|^p dV_\alpha(w)\right)^{1/p}\\
&\leq \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-\lambda_z|^p\, dV_\alpha(w)\right)^{1/p} + \left|\widehat{f_r}(z)-\lambda_z\right|\\
&= \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-\lambda_z|^p\, dV_\alpha(w)\right)^{1/p}\\
&\quad + \left|\frac{1}{|D(z,r)|}\int_{D(z,r)} (f(w)-\lambda_z)\, dV_\alpha(w)\right|\\
&\leq 2\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-\lambda_z|^p\, dV_\alpha(w)\right)^{1/p}\\
&\leq 2C.
\end{aligned}$$

□

Lemma 2.9 *Let $r > 0$ and f be a continuous function on T_B. Then $f \in BO_r$ if and only if there exists a constant C such that*

$$|f(z)-f(w)| \leq C(\beta(z,w)+1)$$

for all $z, w \in T_B$. Furthermore, BO_r is independent of r, so we will simply write BO for BO_r.

Proof It is similar with [17, Lemma 3.3], so we omit it. □

In particular, we also have the following result:

Corollary 2.10 *If $f \in \mathcal{B}$, then*

$$|f(z) - f(w)| \leq \frac{\|f\|_{\mathcal{B}}\beta(z,w)}{\sqrt{2}}$$

for all $z, w \in T_B$.

3 Proofs of the Main Results: Theorems A and B

In this section, we provide the proofs of our main results, **Theorems A and B**, which characterize the relationships between BMO_r^p and properties of holomorphic functions. **Theorem A** states that the following conditions are equivalent:

Theorem 3.1 *Suppose $r > 0$, $1 \leq p < \infty$, and $p(\alpha+1) > \lambda + 1$, let $f \in L^p_\lambda(T_B)$, the following conditions are equivalent:*

(a) $f \in BMO_r^p$.
(b) $f = f_1 + f_2$ with $f_1 \in BO$ and $f_2 \in BA^p$.
(c) $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both bounded on $A^p_\lambda(T_B)$.

Proof $(a) \Rightarrow (b)$: It suffices to show that $BMO_r^p \subset BO + BA^p$. Given $z, w \in T_B$ with $\beta(z,w) < r$, we have

$$\begin{aligned}\left|\widehat{f_r}(z) - \widehat{f_r}(w)\right| &\leq \left|\widehat{f_r}(z) - \widehat{f_{2r}}(z)\right| + \left|\widehat{f_{2r}}(z) - \widehat{f_r}(w)\right| \\ &\leq \frac{1}{|D(z,r)|}\int_{D(z,r)} \left|f(u) - \widehat{f_{2r}}(z)\right| dV_\lambda(u) \\ &\quad + \frac{1}{|D(w,r)|}\int_{D(w,r)} \left|f(v) - \widehat{f_{2r}}(z)\right| dV_\lambda(v).\end{aligned}$$

By Hölder inequality, and Lemma 2.6, so

$$\left|\widehat{f_r}(z) - \widehat{f_r}(w)\right| \leq C\,\|f\|_{BMO_{2r}^p}.$$

It is straightforward to show that $\widehat{f_r}(z)$ is continuous on T_B. Therefore, we have $\widehat{f_r}(z) \in BO$.

Let $g = f - \widehat{f_r}$, we show that $g \in BA^p$. By triangle inequality,

$$\begin{aligned}\left[\widehat{|g|_r^p}(z)\right]^{1/p} &= \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \left|f(u) - \widehat{f_r}(u)\right|^p dV_\lambda(u)\right)^{1/p} \\ &\le \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \left|f(u) - \widehat{f_r}(z)\right|^p dV_\lambda(u)\right)^{1/p} \\ &\quad + \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \left|\widehat{f_r}(z) - \widehat{f_r}(u)\right|^p dV_\lambda(u)\right)^{1/p} \\ &\le \|f\|_{BMO_r^p} + \omega_r(\widehat{f_r})(z) \\ &\le C.\end{aligned}$$

So, $g \in BA^p$, $f = \widehat{f_r} + (f - \widehat{f_r}) \in BO + BA^p$.

$(b) \Rightarrow (c)$: For $f \in BO$, and $g \in A_\lambda^p(T_B)$, from Lemma 2.5, we have

$$\begin{aligned}\left|H_f^{(\alpha)} g(z)\right| &= |(I - P_\alpha)(fg)(z)| \\ &\le \int_{T_B} |f(z) - f(w)|\,|g(w)K_\alpha(z,w)|dV_\alpha(w) \\ &\le C\int_{T_B} \frac{\beta(z,w)+1}{|\boldsymbol{\rho}(z,w)|^{n+1+\alpha}}|g(w)|dV_\alpha(w).\end{aligned}$$

From [17], for any $\varepsilon > 0$, we know that

$$\beta(z,w) \le 2^{2\varepsilon-1}\frac{|\boldsymbol{\rho}(z,w)|^{2\varepsilon}}{\boldsymbol{\rho}(z)^\varepsilon \boldsymbol{\rho}(w)^\varepsilon},$$

then the operator

$$Tg(z) := \int_{T_B} \frac{\beta(z,w)+1}{|\boldsymbol{\rho}(z,w)|^{n+1+\alpha}}|g(w)|dV_\alpha(w)$$

is bounded on $A_\lambda^p(T_B)$, see [16, Lemma 3.4].

Next, for $f \in BA^p$, it follows from [16, Lemma 6.1], let $d\mu_f(z) = |f(z)|^p dV_\lambda(z)$. Given $t \in \mathbb{R}$, we denote by S_t the vector spaces of functions f that are holomorphic in T_B and satisfy the condition:

$$\sup_{z\in T_B} |\boldsymbol{\rho}(z,\mathbf{i})|^t\,|f(z)| < \infty,$$

and it is dense in $A^p_\lambda(T_B)$ when $t > n + 1/2$. So, let $g \in S_t$, we have

$$\begin{aligned}\left\| H^{(\alpha)}_f g \right\|_{A^p_\lambda(T_B)} &= \|fg\|_{A^p_\lambda(T_B)} + \|P_\alpha(fg)\|_{A^p_\lambda(T_B)} \\ &\lesssim \|fg\|_{A^p_\lambda(T_B)} = \int_{T_B} |g(z)|^p \, d\mu_f(z) \\ &\leq C \int_{T_B} |g(z)|^p \, dV_\lambda(z).\end{aligned}$$

Thus, we have $H^{(\alpha)}_f$ and $H^{(\alpha)}_{\overline{f}}$ are both bounded on $A^p_\lambda(T_B)$.

$(c) \Rightarrow (a)$: Inspired by [13], for $f \in L^p_\lambda(T_B)$, and any $z \in T_B$, we can define the function as follows:

$$\mathbf{MO}_\lambda f(z) := \|fh_z - B_\lambda f(z) h_z\|_{L^p_\lambda(T_B)},$$

and

$$f_z(w) = \frac{P_\alpha(\overline{f} h_z)(w)}{h_z(w)}, \ w \in T_B,$$

where $h_z(w) = \frac{K_\lambda(z,w)}{\|K_\lambda(z,\cdot)\|_{A^p_\lambda}}$.

By (2.13), it is easy to know that

$$|K_\lambda(z, w)| \leq 2^{n+1+\lambda} K_\lambda(z, z),$$

and

$$\|K_\lambda(z, \cdot)\|^p_{A^p_\lambda} = \int_{T_B} \frac{\boldsymbol{\rho}(w)^\lambda}{|\boldsymbol{\rho}(z, w)|^{p(n+1+\lambda)}} dV(w) \asymp \boldsymbol{\rho}(z)^{(1-p)(n+1+\lambda)}.$$

Thus, let $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left(\int_{T_B} |f(w) h_z(w)|^p \, dV_\lambda(w) \right)^{1/p} \leq 2^{n+1+\lambda} \|K_\lambda(z, \cdot)\|^{q-1}_{A^p_\lambda} \left(\int_{T_B} |f(w)|^p dV_\lambda(w) \right)^{1/p},$$

which implies that $\overline{f} h_z \in L^p_\lambda(T_B)$. The function f_z and $P_\alpha(\overline{f} h_z)$ are both well-defined, see [16, Lemma 3.4]. Then, similarly to [17, Lemma 4.4], from Lemma 2.5,

we have $\overline{f_z(z)} = B_\lambda f(z)$ and $P_\alpha(\overline{f_z}h_z)(w) = B_\lambda f(z)h_z(w)$. And actually,

$$\begin{aligned}\mathbf{MO}_\lambda^p f(z) &= \|fh_z - B_\lambda f(z)h_z\|_{L_\lambda^p(T_B)}^p \\ &= C\int_{T_B} \frac{|f(w) - B_\lambda f(z)|^p}{|\boldsymbol{\rho}(z,w)|^{p(n+1+\lambda)}} \boldsymbol{\rho}(z)^{(p-1)(n+1+\lambda)} dV_\lambda(w) \\ &\geq C\int_{D(z,r)} \frac{|f(w) - B_\lambda f(z)|^p}{\boldsymbol{\rho}(z)^{n+1+\lambda}} dV_\lambda(w) \\ &\geq \frac{C}{|D(z,r)|}\int_{D(z,r)} |f(w) - B_\alpha f(z)|^p \, dV_\lambda(w),\end{aligned}$$

by Lemma 2.8, if $\mathbf{MO}_\lambda f$ is finite, then $f \in BMO_r^p$.

Using $P_\alpha(\overline{f_z}h_z)(w) = B_\lambda f(z)h_z(w)$, we have

$$\begin{aligned}\mathbf{MO}_\lambda f(z) &= \|fh_z - B_\lambda f(z)h_z\|_{L_\lambda^p(T_B)} \\ &= \left\|fh_z - P_\alpha(\overline{f_z}h_z)\right\|_{L_\lambda^p(T_B)} \\ &\leq \|fh_z - P_\alpha(fh_z)\|_{L_\lambda^p(T_B)} + \left\|P_\alpha(fh_z) - P_\alpha(\overline{f_z}h_z)\right\|_{L_\lambda^p(T_B)} \\ &\leq \left\|H_f^{(\alpha)}h_z\right\|_{L_\lambda^p(T_B)} + \|P_\alpha\|_{L_\lambda^p(T_B)} \left\|\overline{f}h_z - f_z h_z\right\|_{L_\lambda^p(T_B)} \\ &= \left\|H_f^{(\alpha)}h_z\right\|_{L_\lambda^p(T_B)} + \|P_\alpha\|_{L_\lambda^p(T_B)} \left\|\overline{f}h_z - P_\alpha(\overline{f}h_z)\right\|_{L_\lambda^p(T_B)} \\ &= \left\|H_f^{(\alpha)}h_z\right\|_{L_\lambda^p(T_B)} + \|P_\alpha\|_{L_\lambda^p(T_B)} \left\|H_{\overline{f}}^{(\alpha)}h_z\right\|_{L_\lambda^p(T_B)}.\end{aligned}$$

So, if both $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are bounded on $A_\lambda^p(T_B)$, then $f \in BMO_\lambda^p$.

□

Before we prove the second main result, Theorem B, it is necessary to introduce several key lemmas.

Lemma 3.2 *For any fixed $\alpha' \in \mathbb{N}_0^{n-1}$, then we have*

$$\left|(z' - w')^{\alpha'}\right| \lesssim |\boldsymbol{\rho}(z,w)|^{\frac{|\alpha'|}{2}},$$

for all $z, w \in T_B$.

Proof For each fixed $z \in T_B$, we define the following holomorphic self-mapping of T_B:

$$h_z(u) := \left(u' - z', u_n - \mathrm{Re}z_n - iu'\overline{z'} + \frac{i|z'|^2}{2} + \frac{i\overline{z'}\cdot\overline{z'}}{4} + \frac{iu'\cdot z'}{4}\right).$$

All these mappings are holomorphic automorphisms of T_B. In particular, we have

$$h_z(z) = \boldsymbol{\rho}(z)\mathbf{i},$$

where $\mathbf{i} = (0', i)$. Also, an easy calculation shows that

$$\boldsymbol{\rho}(h_z(u), h_z(v)) = \boldsymbol{\rho}(u, v).$$

Note that

$$4|\boldsymbol{\rho}(u, s\mathbf{i})| = |u'^2 - 2iu_n + 2s| \geq |u'|^2,$$

$$\begin{aligned} 4|\boldsymbol{\rho}(u, s\mathbf{i})| \geqslant 4\,|\boldsymbol{\rho}\,(u, \mathbf{i})| \geqslant 4\mathrm{Re}\boldsymbol{\rho}\,(u, \mathbf{i}) &= \sum_{k=1}^{n-1} \mathrm{Re}u_k^2 + 2\mathrm{Im}u_n + 2 \\ &\geqslant \sum_{k=1}^{n-1} \mathrm{Re}u_k^2 + 2\sum_{k=1}^{n-1} (\mathrm{Im}u_k)^2 + 2 \\ &= \sum_{k=1}^{n-1} (\mathrm{Re}u_k)^2 + \sum_{k=1}^{n-1} (\mathrm{Im}u_k)^2 + 2 \\ &= \sum_{k=1}^{n-1} |u_k|^2 + 2, \end{aligned}$$

for any $u \in T_B$ and any $s > 0$. Taking $u = h_z(w)$ and $s = \boldsymbol{\rho}(z)$ in the above inequality, and associating the previous argument, we get

$$\left|(h_z(w))'\right|^2 \lesssim |\boldsymbol{\rho}(h_z(w), \boldsymbol{\rho}(z)\mathbf{i})| = |\boldsymbol{\rho}(h_z(w), h_z(z))| = |\boldsymbol{\rho}(w, z)|.$$

Consequently,

$$\left|(z' - w')^{\alpha'}\right| \lesssim \left|(z' - w')\right|^{|\alpha'|} = \left|(h_z(w))'\right|^{|\alpha'|} \lesssim |\boldsymbol{\rho}(z, w)|^{\frac{|\alpha'|}{2}},$$

as desired. □

Lemma 3.3 *For $f \in H(T_B)$, then*

$$|f(z) - f(w)| \leq \sup_{u\in\gamma} |\widetilde{\nabla} f(u)|\beta(z, w)/\sqrt{2},$$

where γ is a geodesic joining z to w in the Bergman metric.

Proof Let $\beta_{\mathbb{B}_n(\cdot,\cdot)}$ denote the Bergman metric of $\mathbb{B}_n$. From [9] that

$$|h(\xi) - h(\eta)| \leq \sup_{\zeta\in\Gamma} |\widetilde{\nabla}_{\mathbb{B}} h(\zeta)|\beta_{\mathbb{B}_n(\xi,\eta)}$$

for $h \in H(\mathbb{B}_n)$ and $\xi, \eta \in \mathbb{B}_n$, where Γ is a geodesic joining ξ to η with respect to $\beta_{\mathbb{B}_n(\cdot,\cdot)}$. It is known that Φ sends the geodesic of $\mathbb{B}$ with respect to $\beta_{\mathbb{B}_n(\cdot,\cdot)}$ to the

geodesic of T_B with respect to $\beta(\cdot,\cdot)$, and from [12, Proposition 1.4.15],we have the following

$$\beta(z,w)=\beta_{\mathbb{B}}(\Phi^{-1}(z),\Phi^{-1}(w)) \tag{3.1}$$

for any $z,w\in T_B$. Therefore, let $z=\Phi(\xi)$, $w=\Phi(\eta)$, $u=\Phi(\zeta)$ and $\gamma=\Phi(\Gamma)$ be the geodesic joining z to w with respect to $\beta(\cdot,\cdot)$, we have

$$\begin{aligned}|f(z)-f(w)|&=|(f\circ\Phi)(\xi)-(f\circ\Phi)(\eta)|\\&\le\sup_{\zeta\in\Gamma}|\widetilde{\nabla}_{\mathbb{B}}(f\circ\Phi)(\zeta)|\beta_{\mathbb{B}}(\xi,\eta)\\&=\sup_{u\in\gamma}\frac{|\widetilde{\nabla}f(u)|}{\sqrt{2}}\beta(z,w),\end{aligned}$$

where the last equality uses (2.11), as desired. □

Lemma 3.4 *Suppose $r>0$, $\alpha>-1$ and $p>0$. There exists a positive constant C such that*

$$|\widetilde{\nabla}f(z)|^p\le\frac{C}{|D(z,r)|}\int_{D(z,r)}|f(w)|^p dV_\alpha(w)$$

for all $f\in H(T_B)$ and all $z\in T_B$.

Proof From [15, Lemma 2.4], for $g\in H(\mathbb{B}_n)$, there exists a positive constant C depending on r such that

$$|\nabla g(0)|^p\le C\int_{D_{\mathbb{B}_n}(0,r)}|g(\eta)|^p dV_\alpha(\eta).$$

Then, for any $f\in H(T_B)$, we have

$$\begin{aligned}|\nabla(f\circ\Phi)(0)|^p&\le C\int_{D_{\mathbb{B}_n}(0,r)}|f\circ\Phi(\eta)|^p dV_\alpha(\eta)\\&\le C\int_{D(\mathbf{i},r)}\frac{|f(w)|^p}{|\boldsymbol{\rho}(w,\mathbf{i})|^{2(n+1+\alpha)}}dV_\alpha(w)\\&\le C\int_{D(\mathbf{i},r)}|f(w)|^p dV_\alpha(w),\end{aligned}$$

the last inequality uses the inequality (2.13), so $|\boldsymbol{\rho}(w,\mathbf{i})|\ge 1/2$. By (2.10), we have

$$|\widetilde{\nabla}f(\mathbf{i})|^p\le C\int_{D(\mathbf{i},r)}|f(w)|^p dV_\alpha(w).$$

Replacing f by $f \circ \sigma_z^{-1}$ in the above inequality, we obtain

$$|\widetilde{\nabla} f(z)|^p \leq C \int_{D(\mathbf{i},r)} |f \circ \sigma_z^{-1}(w)|^p dV_\alpha(w) = \frac{C}{|D(z,r)|} \int_{D(z,r)} |f(w)|^p dV_\alpha(w),$$

where the last equality uses (2.2) and (2.8). The proof of this lemma is completed. □

We state two key lemmas, Lemma 3.5 and Lemma 3.6, which are crucial for proving **Theorem B**. Using Lemma 3.4 and Lemma 3.3, as discussed above, these new lemmas can be proven straightforwardly using a similar approach to that in [17]. Therefore, we omit the details.

Lemma 3.5 *Suppose $r > 0$, $p \geq 1$, $\alpha > -1$ and $f \in H(T_B)$. Then the following conditions are equivalent:*

(a) $f \in \mathcal{B}$.
(b) There exists a positive constant C such that

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - f(z)|^p dV_\alpha(w) \leq C$$

for all $z \in T_B$.
(c) There exists a positive constant C such that

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - \widehat{f_r}(z)|^p dV_\alpha(w) \leq C$$

for all $z \in T_B$.
(d) There exists a positive constant C with the property that for every $z \in T_B$ there is a complex number c_z such that

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - c_z|^p dV_\alpha(w) \leq C.$$

Lemma 3.6 *Suppose $r > 0$, $p \geq 1$, $\alpha > -1$ and $f \in H(T_B)$. Then the following conditions are equivalent:*

(a) $f \in \mathcal{B}_0$.
(b) There is

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - f(z)|^p dV_\alpha(w) \to 0 \quad as \quad z \to \partial \widehat{T_B}.$$

(c) There is

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - \widehat{f_r}(z)|^p dV_\alpha(w) \to 0 \quad as \quad z \to \partial \widehat{T_B}.$$

(d) For every $z \in T_B$ there is a complex number c_z such that

$$\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-c_z|^p dV_\alpha(w) \to 0 \quad as \quad z \to \partial\widehat{T_B}.$$

And now recall that

$$MO_r(f)(z) = \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w)-\widehat{f_r}(z)|^p dV_\alpha(w)\right)^{1/p},$$

and the space BMO_r^p consists of those functions $f \in L_\alpha^p(T_B)$ such that

$$\|f\|_{BMO_r^p} = \sup\{MO_r(f)(z) : z \in T_B\}.$$

From Lemma 3.5 and 3.6, we conclude that the quantities $\|f\|_{\mathcal{B}}$ and $\|f\|_{BMO_r^p}$ are equivalent. Therefore, **Theorem B** is proved.

Remark 1 The equivalences in Theorem A provide a foundational framework for understanding functions in BMO_r^p, which directly informs the characterization of the Bloch space in Theorem B. By identifying the Bloch space as the intersection of holomorphic functions and BMO_r^p, Theorem B highlights the significance of analytic properties established in Theorem A. Together, these results enhance our understanding of the interrelationships between these function classes and their applications in complex analysis.

4 Proofs of the Main Results: Theorems C and D

To establish that the projection operator $\widetilde{P_\alpha}$ is bounded, we first introduce the integral operator $T_{a,b,\gamma'}$. We will then demonstrate that $T_{a,b,\gamma'}$ is bounded on $L_s^p(T_B)$.

Given $a, b \in \mathbb{R}, \alpha > -1$ and $\gamma' \in \mathbb{N}_0^{n-1}$, we define the integral operator $T_{a,b,\gamma'}$ by

$$T_{a,b,\gamma'}f(z) := \boldsymbol{\rho}(z)^a \int_{T_B} \frac{\left|(z'-w')^{\gamma'}\right|\boldsymbol{\rho}(w)^b}{|\boldsymbol{\rho}(z,w)|^{n+1+a+b+\frac{|\gamma'|}{2}}} f(w)dV_\alpha(w), \quad z \in T_B.$$

Before we proceed with the proof of the theorem, it is important to highlight Lemma 4.1, which serves as a foundational tool in our analysis. We now state this lemma as a crucial step in our argument.

Lemma 4.1 *Suppose that $1 \le p \le \infty$ and $s \in \mathbb{R}$. If $-pa < s+1 < p(b+1)$, then $T_{a,b,\gamma'}$ is bounded on $L_s^p(T_B)$.*

Proof In the special case when $\gamma' = 0'$, this has been shown in [7, Theorem 1]. To prove the general case, by Lemma 3.2, we have

$$|T_{a,b,\gamma'}f(z)| \lesssim T_{a,b,0'}(|f|)(z).$$

□

We now proceed to prove our third result, **Theorem C**.

Theorem 4.2 *$\widetilde{P_\alpha}$ is a bounded projection from $L^\infty(T_B)$ to $\widetilde{\mathcal{B}}$.*

Proof Let $f \in L^\infty(T_B)$. Obviously, $\widetilde{P_\alpha} f(\mathbf{i}) = 0$,

$$\mathcal{L}_j(\widetilde{P_\alpha} f)(z) = \frac{\Gamma(n+2+\alpha)}{2^{n+2}\pi^n\Gamma(\alpha+1)} \int_{T_B} \frac{\overline{w}_j - \overline{z}_j}{\boldsymbol{\rho}(z,w)^{n+2+\alpha}} f(w) dV_\alpha(w),$$

for all $j \in \{1, 2, \cdots, n-1\}$.

By a simple calculation, we can get

$$\boldsymbol{\rho}(z)^{\frac{1}{2}} \left|\mathcal{L}_j(\widetilde{P_\alpha} f)(z)\right| \leq \frac{\Gamma(n+2+\alpha)}{2^{n+2}\pi^n\Gamma(\alpha+1)} \boldsymbol{\rho}(z)^{\frac{1}{2}} \int_{T_B} \frac{\left|\overline{w}_j - \overline{z}_j\right|}{|\boldsymbol{\rho}(z,w)|^{n+2+\alpha}} dV_\alpha(w) \cdot \|f\|_\infty,$$

and

$$\boldsymbol{\rho}(z) \left|\mathcal{L}_n(\widetilde{P_\alpha} f)(z)\right| \leq \frac{\Gamma(n+2+\alpha)}{2^{n+2}\pi^n\Gamma(\alpha+1)} \boldsymbol{\rho}(z) \int_{T_B} \frac{1}{|\boldsymbol{\rho}(z,w)|^{n+2+\alpha}} dV_\alpha(w) \cdot \|f\|_\infty.$$

By applying Lemma 4.1, we obtain $\left\|\widetilde{P_\alpha} f\right\|_{\mathcal{B}} \leq C \|f\|_\infty$, which completes the proof. □

We now proceed to prove the last result, **Theorem D**. First, we introduce the following proposition, which establishes the boundedness of a crucial linear operator.

Proposition 4.3 *Suppose $1 \leq p < \infty$, $t > -1$, for any $\gamma \in \mathbb{N}_0^n$, then the map $f \mapsto \boldsymbol{\rho}^{\langle\gamma\rangle}\mathcal{L}^\gamma f$ is a bounded linear operator from $A_t^p(T_B)$ into $L_t^p(T_B)$. In particular, for any $N \in \mathbb{N}_0$, the map $f \mapsto \mathcal{L}_n^N f$ is a bounded linear operator from $A_t^p(T_B)$ into $A_{t+Np}^p(T_B)$.*

Proof Let λ be sufficiently large such that $p(\lambda+1) > t+1$. Then by Lemma 2.5, we have

$$f(z) = c_\lambda \int_{T_B} \frac{\boldsymbol{\rho}(w)^\lambda}{\boldsymbol{\rho}(z,w)^{n+1+\lambda}} f(w) dV(w),$$

where $c_\lambda = \frac{\Gamma(n+1+\lambda)}{2^{n+1}\pi^n\Gamma(\lambda+1)}$. For any $\gamma \in \mathbb{N}_0^n$,

$$\begin{aligned}
\mathcal{L}^\gamma\left\{\frac{1}{\boldsymbol{\rho}(z,w)^{n+1+\lambda}}\right\} &= C(n,\lambda,\gamma) \frac{\left(\frac{1}{2}(z'-\overline{w'}) - y'i\right)^{\gamma'}}{\boldsymbol{\rho}(z,w)^{n+1+\lambda+|\gamma|}} \\
&= C(n,\lambda,\gamma) \frac{\left(\frac{1}{2}(z'-\overline{z'}+\overline{z'}-\overline{w'}) - y'i\right)^{\gamma'}}{\boldsymbol{\rho}(z,w)^{n+1+\lambda+|\gamma|}} \\
&= C(n,\lambda,\gamma) \frac{\left(\overline{z'}-\overline{w'}\right)^{\gamma'}}{\boldsymbol{\rho}(z,w)^{n+1+\lambda+|\gamma|}}.
\end{aligned}$$

Here, $C(n, \lambda, \gamma)$ is a constant depending on n, λ and γ. Additionally, note that $|\gamma| = \langle\gamma\rangle + |\gamma'|/2$. Therefore,

$$\begin{aligned}\left|\boldsymbol{\rho}^{\langle\gamma\rangle}\mathcal{L}^{\gamma} f(z)\right| &\lesssim \boldsymbol{\rho}(z)^{\langle\gamma\rangle}\int_{T_B}\frac{\left|(z'-w')^{\gamma'}\right|\boldsymbol{\rho}(w)^{\lambda}}{|\boldsymbol{\rho}(z,w)|^{n+1+\lambda+|\gamma|}}|f(w)|dV(w)\\ &= T_{\langle\gamma\rangle,\lambda,\gamma'}(|f|)(z).\end{aligned}$$

The proposition now follows directly from Lemma 4.1. □

Theorem 4.4 *Suppose that* $1 \le p < \infty$, $\lambda > -1$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$\begin{cases}\alpha > \frac{\lambda+1}{p} - 1, & 1 < p < \infty,\\ \alpha \ge \lambda, & p = 1.\end{cases}$$

If $f \in A^p_\lambda(T_B)$ *then*

$$f = \frac{(2i)^N\Gamma(1+\alpha)}{\Gamma(1+\alpha+N)}\mathcal{T}_\alpha(\boldsymbol{\rho}^N\mathcal{L}^N_n f)$$

for any $N \in \mathbb{N}_0$.

To prove this theorem, we divide the proof into two separate lemmas, each addressing a distinct case. We begin by considering the case where $p = 1$ and $\lambda = \alpha$.

Lemma 4.5 *Theorem 4.4 holds for* $p = 1$ *with* $\lambda = \alpha$.

Proof Suppose that $f \in A^1_\alpha(T_B)$. Let $\gamma > \alpha$, by Lemma 2.5, we have

$$f(z) = c_\gamma\int_{T_B}\frac{\boldsymbol{\rho}(w)^{\gamma}}{\boldsymbol{\rho}(z,w)^{n+1+\gamma}}f(w)dV(w)$$

for all $z \in T_B$. Then it is easy to shows that

$$\mathcal{L}^N_n f(z) = (n+1+\gamma)_N(-i/2)^N c_\gamma\int_{T_B}\frac{\boldsymbol{\rho}(w)^{\gamma}}{\boldsymbol{\rho}(z,w)^{n+1+\gamma+N}}f(w)dV(w),$$

where $(n+1+\gamma)_N$ is the Pochhammer symbol denoted by $(a)_k = a(a+1)\cdots(a+k-1)$. Thus,

$$\begin{aligned}\mathcal{T}_\alpha(\boldsymbol{\rho}^N\mathcal{L}^N_n f)(z) &= c_\alpha\int_{T_B}\frac{\boldsymbol{\rho}(w)^{\alpha+N}\mathcal{L}^N_n f(w)}{\boldsymbol{\rho}(z,w)^{n+1+\alpha}}dV(w)\\ &= c_\alpha(n+1+\gamma)_N(-i/2)^N c_\gamma\\ &\quad\times\int_{T_B}\frac{\boldsymbol{\rho}(w)^{\alpha+N}}{\boldsymbol{\rho}(z,w)^{n+1+\alpha}}\left(\int_{T_B}\frac{\boldsymbol{\rho}(u)^{\gamma}}{\boldsymbol{\rho}(w,u)^{n+1+\gamma+N}}f(u)dV(u)\right)dV(w).\end{aligned}$$

By Fubini's theorem and Lemma 2.4, for any $z \in T_B$, the above double integral equals to

$$\begin{aligned}
&\int_{T_B} \boldsymbol{\rho}(u)^{\gamma} f(u) \left(\int_{T_B} \frac{\boldsymbol{\rho}(w)^{\alpha+N}}{\boldsymbol{\rho}(z,w)^{n+1+\alpha} \boldsymbol{\rho}(w,u)^{n+1+\gamma+N}} dV(w) \right) dV(u) \\
&\quad = C_1(n, n+1+\alpha, n+1+\gamma+N, \lambda+N) \int_{T_B} \frac{\boldsymbol{\rho}(u)^{\gamma} f(u)}{\boldsymbol{\rho}(z,u)^{n+1+\gamma}} dV(u) \\
&\quad = C_1(n, n+1+\alpha, n+1+\gamma+N, \alpha+N) c_{\gamma}^{-1} f(z).
\end{aligned}$$

Consequently,

$$\mathcal{T}_{\alpha} \left(\boldsymbol{\rho}^N \mathcal{L}_n^N f \right)(z) = \frac{(-i/2)^N \Gamma(1+\alpha+N)}{\Gamma(1+\alpha)} f(z).$$

We now justify the use of Fubini's theorem. For any fixed $z \in T_B$, by Lemma 2.4 (note that $\gamma > \alpha$) and (2.13), it follows that

$$\begin{aligned}
&\int_{T_B} \left(\int_{T_B} \frac{\boldsymbol{\rho}(w)^{\alpha+N}}{|\boldsymbol{\rho}(z,w)|^{n+1+\alpha} |\boldsymbol{\rho}(w,u)|^{n+1+\gamma+N}} dV(w) \right) \boldsymbol{\rho}(u)^{\gamma} |f(u)| dV(u) \\
&\leq (\boldsymbol{\rho}(z)/2)^{-n-1-\alpha} \int_{T_B} \left(\int_{T_B} \frac{\boldsymbol{\rho}(w)^{\alpha+N}}{|\boldsymbol{\rho}(w,u)|^{n+1+\gamma+N}} dV(w) \right) \boldsymbol{\rho}(u)^{\gamma} |f(u)| dV(u) \\
&= (\boldsymbol{\rho}(z)/2)^{-n-1-\alpha} C_1(n, n+1+\gamma+N, \alpha+N) \int_{T_B} |f(u)| \boldsymbol{\rho}(u)^{\alpha} dV(u) < \infty.
\end{aligned}$$

The proof of the lemma is completed. □

By applying Lemma 4.6, we can now address the remaining part of the proof.

Lemma 4.6 *Theorem 4.4 holds for $1 < p < \infty$ when $p(\alpha+1) > \lambda+1$, and for $p = 1$ when $\alpha > \lambda$.*

Proof Put

$$g := f - \frac{(2i)^N \Gamma(1+\alpha)}{\Gamma(1+\alpha+N)} \mathcal{T}_{\alpha} \left(\boldsymbol{\rho}^N \mathcal{L}_n^N f \right).$$

Note from Proposition 4.3 that $\mathcal{L}_n^N f \in A^p_{\lambda+Np}(T_B)$. It is not difficult to see that $\mathcal{T}_{\alpha}(\boldsymbol{\rho}^N \mathcal{L}_n^N f)$ belongs to $A^p_{\lambda}(T_B)$ and so does g. A simple calculation shows that

$$\begin{aligned}
\mathcal{L}_n^N g(z) &= \mathcal{L}_n^N f(z) - c_{\alpha+N} \int_{T_B} \frac{\boldsymbol{\rho}(w)^{\alpha+N}}{\boldsymbol{\rho}(z,w)^{n+1+\alpha+N}} \mathcal{L}_n^N f(w) dV(w) \\
&= \mathcal{L}_n^N f(z) - \mathcal{T}_{\alpha+N} \left(\mathcal{L}_n^N f \right)(z) = 0.
\end{aligned} \tag{4.1}$$

By [18], since $p(\alpha + N + 1) > \lambda + Np + 1$ for any case.

Suppose $\mathcal{L}_n^N g \equiv 0$. Then g has the form

$$g(z) = g_{N-1}(z')z_n^{N-1} + g_{N-2}(z')z_n^{N-2} + \cdots + g_0(z')$$

with $g_{N-1}, \cdots, g_0$ holomorphic functions of z'. It follows that

$$\mathcal{L}_n^{N-1} g(z) = (N-1)! g_{N-1}\left(z'\right).$$

All that remains is to show that $g_{N-1}(z') \equiv 0$. In view of Proposition 4.3, we know that $\boldsymbol{\rho}^{N-1}\mathcal{L}_n^{N-1} g \in L_\lambda^p(T_B)$. Thus,

$$\begin{aligned}\infty > & \int_{T_B} \boldsymbol{\rho}(z)^{(N-1)p+\lambda} \left|g_{N-1}(z')\right|^p dV(z) \\ = & \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \int_{y_n > y'^2} (y_n - y'^2)^{(N-1)p+\lambda} dy_n |g_{N-1}(x' + iy')|^p dy' dx.\end{aligned}$$

However, note that the inner integral diverges for any fixed y', implying that $g_{N-1}(z') \equiv 0$. Consequently, after a finite number of iterations, we conclude that $g \equiv 0$, thus completing the proof. □

By combining the two previously discussed lemmas, we have successfully completed the proof of Theorem D.

Author Contributions D: Proposed the main ideas, extended the results to tubular domains, proved the main theorems, and wrote the initial draft of the manuscript. L: Helped Jiaqing Ding overcome difficulties in proving the theorems, guided the completion of the proofs, and revised the final manuscript. F: Participated in discussions, provided valuable suggestions, helped verify the theorems, and assisted with revising the manuscript. Z: Reviewed the final manuscript, provided feedback, and refined the details of the proofs.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they no conflict of interest.

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李海绸　先生/女士：

根据《国家自然科学基金条例》、相关项目管理办法规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12326407，项目名称：2024复分析及其应用专题讲习班 ，直接费用：20.00万元，项目起止年月：2024年01月至 2024年 12月，有关项目的评审意见及修改意见附后。

请您尽快登录科学基金网络信息系统（https://grants.nsfc.gov.cn），**认真阅读《国家自然科学基金资助项目计划书填报说明》并按要求填写《国家自然科学基金资助项目计划书》（以下简称计划书）**。对于有修改意见的项目，请您按修改意见及时调整计划书相关内容；如您对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

请您将电子版计划书通过科学基金网络信息系统（https://grants.nsfc.gov.cn）提交，由依托单位审核后提交至自然科学基金委。自然科学基金委审核未通过者，将退回的电子版计划书修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印）并在项目负责人承诺栏签字，由依托单位科研、财务管理等部门审核、签章并在承诺栏加盖依托单位公章，且将申请书纸质签字盖章页订在其中一份计划书之后，一并报送至自然科学基金委项目材料接收工作组。纸质版计划书应当保证与审核通过的电子版计划书内容一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委提交电子版计划书、报送纸质版计划书并补交申请书纸质签字盖章页截止时间节点如下：

1. **2024年01月22日16点：**提交电子版计划书的截止时间；

2. **2024年01月29日16点：**提交修改后电子版计划书的截止时间；

3. **2024年02月05日：**报送纸质版计划书（一式两份，其中一份包含申请书纸质签字盖章页）的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，逾期不报计划书或申请书纸质签字盖章页且未说明理由的，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2024年01月02日

国家自然科学基金

资助项目准予结题通知

李海绸　同志：

您承担的国家自然科学基金项目：(2024复分析及其应用专题讲习班)，批准号：(12326407)按有关规定已审核完毕，准予结题。

与本项目资助有关的后续成果，请您继续及时报送。

祝您在研究工作中取得更好的成绩！

国家自然科学基金委员会

数理科学部

2025年3月31日